Complex Analysis

Several Complex Variables and Connections with PDE Theory and Geometry

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Peter Ebenfelt
Norbert Hungerbühler
Joseph J. Kohn
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Emil J. Straube
Editors

Trends in Mathematics

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Preface

Norbert Hungerbühler

The idea to organize a conference in honour of Linda Rothschild emerged in 2006. This idea began to substantiate in 2007 when the Swiss Mathematical Society assigned the traditional Spring Meeting to the University of Fribourg. An organizing committee was quickly formed:

Organizing committee

Norbert Hungerbühler	University of Fribourg, Switzerland
Frank Kutzschebauch	University of Berne, Switzerland
Bernhard Lamel	University of Vienna, Austria
Francine Meylan	University of Fribourg, Switzerland
Nordine Mir	Université de Rouen, France

In order to ensure a high-quality conference program, the search for a scientific committee began. Soon after, a distinguished group was found who started working right away:

Scientific committee

Peter Ebenfelt	University of California, San Diego, USA
Franc Forstnerič	University of Ljubljana, Slovenia
Joseph J. Kohn	Princeton University, USA
Emil J. Straube	Texas A&M University, USA

Spring Meeting of the Swiss Mathematical Society
Conference on Complex Analysis 2008
Several Complex Variables and Connections with PDEs and Geometry
In honour of Linda Rothschild, Fribourg, July 7–11

Only a little while later it became clear that the subject and the top-class speakers who agreed to participate in the conference called for a proceedings volume to make the presented results available shortly after the conference. This project was carried out under the direction of the editorial board:

Focus on youth

The aim of the conference was to gather worldwide leading scientists, and to offer the occasion to PhD students and postdocs to come into contact with them. The committees explicitly encouraged young scientists, doctoral students and postdocs to initiate scientific contact and to aim at an academic career. The topic of the conference was apparently very attractive for young scientists, and the event an ideal platform to promote national and international doctoral students and postdocs. This aspect became manifest in a poster session where junior researchers presented their results.

The conference was intended to have a strong component in instruction of PhD students: Three mini courses with introductory character were held by Pengfei Guan, Mei-Chi Shaw and Ngaiming Mok. These three mini courses have been very well received by a large audience and were framed by the series of plenary lectures presenting newest results and techniques.

The participation of junior female researchers, PhD students and mathematicians from developing countries has been encouraged in addition by offering grants for traveling and accommodation.

The subject

The conference *Complex Analysis 2008* has been devoted to the subject of *Several Complex Variables and Connections with PDEs and Geometry.* These three main subject areas of the conference have shown their deep relations, and how techniques from each of these fields can influence the others. The conference has stimulated further interaction between these areas.

The conference was held in honor of Prof. Linda Rothschild who is one of the most influential contributors of the subject during the last decades. A particular aim was to encourage female students to pursue an academic career. In fact, female mathematicians have been well represented among the speakers, in the organizing committee and in the poster sessions.

Several Complex Variables is a beautiful example of a field requiring a wide range of techniques coming from diverse areas in Mathematics. In the last decades, many major breakthroughs depended in particular on methods coming from Partial Differential Equations and Differential and Algebraic Geometry. In turn, Several Complex Variables provided results and insights which have been of fundamental importance to these fields. This is in particular exemplified by the subject of Cauchy-Riemann geometry, which concerns itself both with the tangential Cauchy-Riemann equations and the unique mixture of real and complex geometry that real objects in a complex space enjoy. CR geometry blends techniques from algebraic geometry, contact geometry, complex analysis and PDEs; as a unique meeting point for some of these subjects, it shows evidence of the possible synergies of a fusion of the techniques from these fields.

The interplay between PDE and Complex Analysis has its roots in Hans Lewy's famous example of a locally non solvable PDE. More recent work on PDE has been similarly inspired by examples from CR geometry. The application of analytic techniques in algebraic geometry has a long history; especially in recent years, the analysis of the $\bar{\partial}$-operator has been a crucial tool in this field. The $\bar{\partial}$-operator remains one of the most important examples of a partial differential operator for which regularity of solutions under boundary constraints have been extensively studied. In that respect, CR geometry as well as algebraic geometry have helped to understand the subtle aspects of the problem, which is still at the heart of current research.

Summarizing, our conference has brought together leading researchers at the intersection of these fields, and offered a platform to discuss the most recent developments and to encourage further interactions between these mathematicians. It was also a unique opportunity for younger people to get acquainted with the current research problems of these areas.

Organization

The conference was at the same time the 2008 Spring Meeting of the Swiss Mathematical Society. The event has profited from the organizational structures of the SMS and the embedding in the mathematical community of Switzerland. The University of Fribourg has proven to be the appropriate place for this international event because of its tradition in Complex Analysis, the central geographic location, and its adequate infrastructure. In turn, its reputation and that of the region has benefited from this conference.

The conference has been announced internationally in the most important conference calendars and in several journals. Moreover, the event has been advertised by posters in numerous mathematics institutes worldwide, by e-mails and in the regular announcements of the Swiss Mathematical Society.

Acknowledgment

It becomes increasingly difficult to find sponsors for conferences of the given size, in particular in mathematics. We are all the more grateful to our sponsors who have generously supported the conference, and the proceedings volume in hand:

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- Rectorate, University of Fribourg
- Swiss National Science Foundation
- Walter Haefner Stiftung
- Swiss Doctoral Program in Mathematics

In the name of the conference committees and of all participants, we would like to thank all sponsors – foundations, institutions and companies – very cordially for their contributions and the shown appreciation for our work as mathematicians: Thank you!

We also thank the team of Dr. Thomas Hempfling of the Birkhäuser publishing company for their help and professional expertise during the production process of these proceedings.

Finally, we would like to thank Elisabeth François and Claudia Kolly who assumed the secretariat of the conference.

Fribourg, August 2009 Norbert Hungerbühler

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Extended Curriculum Vitae of Linda Preiss Rothschild

Linda Rothschild was born February 28, 1945, in Philadelphia, PA. She received her undergraduate degree, magna cum laude, from the University of Pennsylvania in 1966 and her PhD in mathematics from MIT in 1970. Her PhD thesis was "On the Adjoint Action of a Real Semisimple Lie Group". She held positions at Tufts University, Columbia University, the Institute for Advanced Study, and Princeton University before being appointed an associate professor of mathematics at the University of Wisconsin-Madison in 1976. She was promoted to full professor in 1979. Since 1983 she has been professor of mathematics at the University of California at San Diego, where she is now a Distinguished Professor.

Rothschild has worked in the areas of Lie groups, partial differential equations and harmonic analysis, and the analytic and geometric aspects of several complex variables. She has published over 80 papers in these areas. Rothschild was awarded an Alfred P. Sloan Fellowship in 1976. In 2003 she won the Stefan Bergman Prize from the American Mathematical Society (jointly with Salah Baouendi). The citation read in part:

> *"The Bergman Prize was awarded to Professors Salah Baouendi and Linda Rothschild for their joint and individual work in complex analysis. In addition to many important contributions to complex analysis they have also done first rate work in the theory of partial differential equations. Their recent work is centered on the study of CR manifolds to which they and their collaborators have made fundamental contributions.*
>
> *Rothschild, in a joint paper with E. Stein, introduced Lie group methods to prove L^p and Hölder estimates for the sum of squares operators as well as the boundary Kohn Laplacian for real hypersurfaces. In later joint work with L. Corwin and B. Helfer, she proved analytic hypoellipticity for a class of first-order systems. She also proved the existence of a family of weakly pseudoconvex hypersurfaces for which the boundary Kohn Laplacian is hypoelliptic but does not satisfy maximal L^2 estimates."*

In 2005, Rothschild was elected a Fellow of the American Academy of Arts and Sciences, and in 2006 she was an invited speaker at the International Congress of Mathematics in Madrid.

Rothschild served as President of the Association for Women in Mathematics from 1983 to 1985 and as Vice-President of the American Mathematical Society from 1985 to 1987. She served on the editorial committees of the Transactions of the AMS and Contemporary Mathematics. She is also an editorial board member of Communications in Partial Differential Equations and co-founder and co-editor-in-chief of Mathematical Research Letters. She has served on many professional committees, including several AMS committees, NSF panels, and an organization committee for the Special Year in Several Complex Variables at the Mathematical Sciences Research Institute. She presented the 1997 Emmy Noether Lecture for the AWM. Rothschild has a keen interest in encouraging young women who want to study mathematics. A few years ago she helped establish a scholarship for unusually talented junior high school girls to accelerate their mathematical training by participating in a summer program.

Educational Background

B.A.	University of Pennsylvania, 1966
Ph.D.	in mathematics, Massachusetts Institute of Technology, 1970 Dissertation: *On the Adjoint Action of a Real Semisimple Lie Group* Advisor: Isadore Manual Singer

Professional Employment

1982–	Professor, University of California, San Diego
2001–05	Vice Chair for Graduate Affairs, Mathematics Dept., UCSD
1979–82	Professor, University of Wisconsin
1981–82	Member, Institute for Advanced Study
1978	Member, Institute for Advanced Study
1976–77	Associate Professor, University of Wisconsin
1975–76	Visiting Assistant Professor, Princeton University
1974–75	Member, Institute for Advanced Study
1972–74	Ritt Assistant Professor, Columbia University
1970–72	Assistant Professor, Tufts University
1970–72	Research Staff, Artificial Intelligence Laboratory, M.I.T.

Honors and Fellowships

2005	Fellow, American Academy of Arts and Sciences
2003	Stefan Bergman Prize
1976–80	Alfred P. Sloan Foundation Fellow
1966–70	National Science Foundation Graduate Fellow

Selected Invited Lectures

- Invited address, International Congress of Mathematicians, Madrid, August 2006
- "Frontiers in Mathematics" Lecturer, Texas A&M University, September 1999
- Invited hour speaker, Sectional joint meeting of American Mathematical Society and Mathematical Association of America, Claremont, October 1997
- Emmy Noether Lecturer (Association for Women in Mathematics), Annual Joint Mathematics Meetings, San Diego January 1997
- Invited hour lecturer, Annual Joint Mathematics Meetings, Orlando, January 1996
- Invited hour speaker, Annual Summer meeting of American Mathematics Society, Pittsburgh, August1981

Students

Mark Marson	University of California, San Diego,	1990
Joseph Nowak	University of California, San Diego,	1994
John Eggers	University of California, San Diego,	1995
Bernhard Lamel	University of California, San Diego,	2000
Slobodan Kojcinovic	University of California, San Diego,	2001
Robert Kowalski	University of California, San Diego,	2002

Selected National Committees and Offices

National Science Foundation, Mathematics Division

- Advisory Panel, 1984–87 and other panels 1997–99, 2004

American Mathematical Society (AMS)

- Bocher Prize Committee 2001–04
- National Program Committee 1997–2000
  Chair 1998–1999
- Nominating Committee, 1982–84, 1994–96
- Committee on Science Policy, 1979–82, 92–9
- AMS Vice President, 1985–87
- Committee on Committees, 1977–79, 1979–81
- Executive Committee, 1978–80
- Council of the AMS, 1977–80

Association for Women in Mathematics (AWM)

- Noether Lecture Committee 1988–90, 1994–1997
  Chair 1989–90
- Schafer Prize Committee 1993–94
- AWM President, 1983–85.

Mathematical Association of America

- Chauvenet Prize Committee, 1998–2000

Mathematical Sciences Research Institute

- Board of Trustees, 1996–1999
- Budget Committee 1996–1998

California Science Museum

- Jury to select California Scientist of the Year Award, 1995–1999

Institute for Pure and Applied Mathematics (IPAM)

- Board of Trustees, 2002–2005

Editorial Positions

- Co-Editor-in-Chief, Mathematical Research Letters, 1994–
- Editorial Board, Journal of Mathematical Analysis and Applications, 2001–
- Editorial Board, Communications in Partial Differential Equations, 1984–
- Editorial Board, Contemporary Mathematics, 1990–1994
- Editor for complex and harmonic analysis, Transactions of the American Mathematical Society, 1983–1986

Publication List of Linda Preiss Rothschild

[1] Peter Ebenfelt and Linda P. Rothschild. New invariants of CR manifolds and a criterion for finite mappings to be diffeomorphic. *Complex Var. Elliptic Equ.*, 54(3-4):409–423, 2009. ISSN 1747-6933.

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[3] Peter Ebenfelt and Linda P. Rothschild. Analyticity of smooth CR mappings of generic submanifolds. *Asian J. Math.*, 11(2):305–318, 2007. ISSN 1093-6106.

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Oblique Polar Lines of $\int_X |f|^{2\lambda}|g|^{2\mu}\square$

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Abstract. Existence of oblique polar lines for the meromorphic extension of the current valued function $\int |f|^{2\lambda}|g|^{2\mu}\square$ is given under the following hypotheses: f and g are holomorphic function germs in $\mathbb{C}^{n+1}$ such that g is non-singular, the germ $\Sigma := \{\,\mathrm{d}f \wedge \mathrm{d}g = 0\}$ is one dimensional, and g is proper and finite on $S := \{df = 0\}$. The main tools we use are interaction of strata for f (see [4]), monodromy of the local system $H^{n-1}(u)$ on S for a given eigenvalue $\exp(-2i\pi u)$ of the monodromy of f, and the monodromy of the cover $g|_S$. Two non-trivial examples are completely worked out.

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Introduction

Given an open subset Y in $\mathbb{C}^m$, two holomorphic functions f, g on Y and a $\mathcal{C}^\infty$ compactly support (m,m)-form ϕ in Y, the integral $\int_Y |f|^{2\lambda}|g|^{2\mu}\phi$, for (λ,μ) in $\mathbb{C}^2$ with $\Re\lambda$ and $\Re\mu > 0$, defines a holomorphic function in that region. As a direct consequence of the resolution of singularities, this holomorphic function extends meromorphically to $\mathbb{C}^2$, see Theorem 1.1. The polar locus of this extension is contained in a union of straight lines with rational slopes (see [8] for other results on this integral). In this paper we look for geometric conditions that guarantee a true polar line of this extension for at least one $\phi \in \Lambda^{m,m}C_c^\infty(Y)$, in other words a true polar line of the meromorphic extension of the holomorphic current valued function

$$(\lambda,\mu) \mapsto \int_Y |f|^{2\lambda}|g|^{2\mu}\square.$$

Since existence of horizontal or vertical polar lines follows directly from existence of poles of $\int_Y |g|^{2\mu}\square$ or $\int_Y |f|^{2\lambda}\square$ that have been extensively studied in [1], [2] and [3], we will concentrate on oblique polar lines. Because desingularization is quite hard to compute, it is not clear how to determine these polar lines. Moreover, only a few of the so obtained candidates are effectively polar and no geometric conditions are known to decide it in general.

In Section 2, we expose elementary properties of meromorphic functions of two variables that are used later for detecting oblique polar lines. Four examples of couples (f, g) for which these results apply are given.

In Sections 3 and 4 we give sufficient criteria to obtain oblique polar lines in rather special cases, but with a method promised to a large generalization. They rely on results which give realization in term of holomorphic differential forms of suitable multivalued sections of the sheaf of vanishing cycles along the smooth part of the singular set S (assumed to be a curve) of the function f. The second function g being smooth and transversal to S at the origin. The sufficient condition is then given in term of the monodromy on $S^* := S \setminus \{0\}$ on the sheaf of vanishing cycles of f for the eigenvalue $\exp(-2i\pi u)$ assuming that the meromorphic extension of $\int_X |f|^{2\lambda}\square$ has only simple poles at $-u-q$ for all $q \in \mathbb{N}$ (see Corollary 4.3).

To be more explicit, recall the study of $\int |f|^{2\lambda}\square$ started in [4] and completed in [5], for a holomorphic function f defined in an open neighbourhood of $0 \in \mathbb{C}^{n+1}$ with one-dimensional critical locus S. The main tool was to restrict f to hyperplane sections transverse to S^* and examine, for a given eigenvalue $\exp(-2i\pi u)$ of the monodromy of f, the local system $H^{n-1}(u)$ on S^* formed by the corresponding spectral subspaces. Higher-order poles of the current valued meromorphic function $\int |f|^{2\lambda}\square$ at $-u-m$, some $m \in \mathbb{N}$, are detected using the existence of a uniform section of the sheaf $H^{n-1}(u)$ on S^* which is not extendable at the origin. So an important part of this local system remained unexplored in [4] and [5] because only the eigenvalue 1 of the monodromy Θ of the local system $H^{n-1}(u)$ on S^* is involved in the exact sequence

$$0 \to H^0(S, H^{n-1}(u)) \to H^0(S^*, H^{n-1}(u)) \to H^1_{\{0\}}(S, H^{n-1}(u)) \to 0.$$

In this paper, we will focus on the other eigenvalues of Θ.

Let us assume the following properties:

(1) the function g is non-singular near 0;
(2) the set $\Sigma := \{\, \mathrm{d}f \wedge \mathrm{d}g = 0\}$ is a curve;
(3) the restriction $g|_S : S \to \mathbb{D}$ is proper and finite;
(4) $g|_S^{-1}(0) = \{0\}$ and $g|_{S^*}$ is a finite cover of $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$.

Condition (2) implies that the singular set $S := \{\, \mathrm{d}f = 0\}$ of f has dimension $\leqslant 1$. We are interested in the case where S is a curve.

Remark that condition (4) may always be achieved by localization near 0 when conditions (1), (2) and (3) are satisfied. These conditions hold in a neighbourhood of the origin if (f, g) forms an isolated complete intersection singularity (icis) with one-dimensional critical locus, assuming g smooth. But we allow also the case where Σ has branches in $\{f = 0\}$ not contained in S.

The direct image by g of the constructible sheaf $H^{n-1}(u)$ supported in S will be denoted by $\mathcal{H}$; it is a local system on $\mathbb{D}^*$. Let $\mathcal{H}_0$ be the fibre of $\mathcal{H}$ at $t_0 \in \mathbb{D}^*$ and Θ_0 its monodromy which is an automorphism of $\mathcal{H}_0$. In case where S is smooth, it is possible to choose the function g in order that $g|_S$ is an isomorphism and Θ_0 may be identified with the monodromy Θ of $H^{n-1}(u)$ on S^*. In general, Θ_0 combines Θ and the monodromy of the cover $g|_{S^*}$.

Take an eigenvalue[1] $\exp(-2i\pi l/k) \neq 1$ of Θ, with $l \in [1, k-1]$ and $(l,k) = 1$. We define an analogue of the interaction of strata in this new context. The auxiliary non singular function g is used to realize analytically the rank one local system on S^* with monodromy $\exp(-2i\pi l/k)$. To perform this we shall assume that the degree of g on the irreducible branch of S we are interested in, is relatively prime to k. Of course this is the case when S is smooth and g transversal to S at the origin. Using then a kth root of g we can lift our situation to the case where we consider an invariant section of the complex of vanishing cycles of the lifted function $\tilde{f}$ (see Theorem 4.2) and then use already known results from [4]. The existence of true oblique polar lines follows now from results of Section 2.

The paper ends with a complete computation of two non-trivial examples that illustrate the above constructions.

1. Polar structure of $\int_X |f|^{2\lambda}\,\square$

Theorem 1.1. Bernstein & Gelfand. *For m and $p \in \mathbb{N}^*$, let Y be an open subset in $\mathbb{C}^m$, $f : Y \to \mathbb{C}^p$ a holomorphic map and X a relatively compact open set in Y. Then there exists a finite set $P(f) \subset \mathbb{N}^p \setminus \{0\}$ such that, for any form $\phi \in \Lambda^{m,m}C_c^\infty(X)$ with compact support, the holomorphic map in the open set $\{\Re\lambda_1 > 0\} \times \cdots \times \{\Re\lambda_p > 0\}$ given by*

$$(\lambda_1, \ldots, \lambda_p) \mapsto \int_X |f_1|^{2\lambda_1} \ldots |f_p|^{2\lambda_p}\phi \tag{1.1}$$

has a meromorphic extension to $\mathbb{C}^p$ with poles contained in the set

$$\bigcup_{a \in P(f), l \in \mathbb{N}^*} \{\langle a \mid \lambda\rangle + l = 0\}.$$

Proof. For sake of completeness we recall the arguments of [10].

Using desingularization of the product $f_1 \ldots f_p$, we know [12] that there exists a holomorphic manifold $\tilde{Y}$ of dimension m and a holomorphic proper map $\pi : \tilde{Y} \to Y$ such that the composite functions $\tilde{f}_j := f_j \circ \pi$ are locally expressible as

$$\tilde{f}_k(y) = y_1^{a_1^k} \ldots y_m^{a_m^k} u_k(y), 1 \leqslant k \leqslant p, \tag{1.2}$$

where $a_j^k \in \mathbb{N}$ and u_k is a holomorphic nowhere vanishing function. Because $\pi^{-1}(X)$ is relatively compact, it may be covered by a finite number of open set where (1.2) is valid.

For $\varphi \in \Lambda^{m,m}C_c^\infty(X)$ and $\Re\lambda_1, \ldots, \Re\lambda_p$ positive, we have

$$\int_X |f_1|^{2\lambda_1} \ldots |f_p|^{2\lambda_p}\phi = \int_{\pi^{-1}(X)} |\tilde{f}_1|^{2\lambda_1} \ldots |\tilde{f}_p|^{2\lambda_p}\pi^*\phi.$$

[1]Note that the eigenvalues of Θ are roots of unity.

Using partition of unity and setting $\mu_k := a_k^1\lambda_1 + \cdots + a_k^p\lambda_p$, $1 \leqslant k \leqslant m$, we are reduced to give a meromorphic extension to

$$(\mu_1, \ldots, \mu_m) \to \int_{\mathbb{C}^m} |y_1|^{2\mu_1} \ldots |y_m|^{2\mu_m} \omega(\mu, y), \tag{1.3}$$

where ω is a C^∞ form of type (m,m) with compact support in $\mathbb{C}^m$ valued in the space of entire functions on $\mathbb{C}^m$. Of course, (1.3) is holomorphic in the set $\{\Re\mu_1 > -1, \ldots, \Re\mu_m > -1\}$.

The relation

$$(\mu_1 + 1).|y_1|^{2\mu_1} = \partial_1(|y_1|^{2\mu_1}.y_1)$$

implies by partial integration in y_1

$$\int_{\mathbb{C}^m} |y_1|^{2\mu_1} \ldots |y_m|^{2\mu_m} \omega(\mu, y) = \frac{-1}{\mu_1 + 1} \int_{\mathbb{C}^m} |y_1|^{2\mu_1}.y_1.|y_2|^{2\mu_2} \ldots |y_m|^{2\mu_m} \partial_1\omega(\mu, y).$$

Because $\partial_1\omega$ is again a C^∞ form of type (m,m) with compact support in $\mathbb{C}^m$ valued in the space of entire functions on $\mathbb{C}^m$, we may repeat this argument for each coordinate $y_2, \ldots, y_m$ and obtain

$$\int_{\mathbb{C}^m} |y_1|^{2\mu_1} \ldots |y_m|^{2\mu_m} \omega(\mu, y) =$$
$$= \frac{(-1)^m}{(\mu_1 + 1) \ldots (\mu_m + 1)} \int_{\mathbb{C}^m} |y_1|^{2\mu_1}.y_1.|y_2|^{2\mu_2}.y_2 \ldots |y_m|^{2\mu_m}.y_m.\partial_1 \ldots \partial_m\omega(\mu, y).$$

The integral on the RHS is holomorphic for $\Re\mu_1 > -3/2, \ldots, \Re\mu_m > -3/2$. Therefore the function (1.3) is meromorphic in this domain with only possible poles in the union of the hyperplanes $\{\mu_1 + 1 = 0\}, \ldots, \{\mu_m + 1 = 0\}$.

Iteration of these arguments concludes the proof. □

Remark 1.2. An alternate proof of Theorem 1.1 has been given for $p = 1$ by Bernstein [9], Björk [11], Barlet-Maire [6], [7]. See also Loeser [13] and Sabbah [14] for the general case.

In case where $f_1, \ldots, f_p$ define an isolated complete intersection singularity (icis), Loeser and Sabbah gave moreover the following information on the set $P(f)$ of the "slopes" of the polar hyperplanes in the meromorphic extension of the function (1.1): it is contained in the set of slopes of the discriminant locus Δ of f, which in this case is an hypersurface in $\mathbb{C}^p$. More precisely, take the $(p-1)$-skeleton of the fan associated to the Newton polyhedron of Δ at 0 and denote by $P(\Delta)$ the set of directions associated to this $(p-1)$-skeleton union with the set $\{(a_1, \ldots, a_p) \in \mathbb{N}^p \mid a_1 \ldots a_p = 0\}$. Then

$$P(f) \subseteq P(\Delta).$$

In particular, if the discriminant locus is contained in the hyperplanes of coordinates, then there are no polar hyperplanes with direction in $(\mathbb{N}^*)^p$.

The results of Loeser and Sabbah above have the following consequence for an icis which is proved below directly by elementary arguments, after we have introduced some terminology.

Definition 1.3. Let $f_1, \ldots, f_p$ be holomorphic functions on an open neighbourhood X of the origin in $\mathbb{C}^m$. We shall say that a polar hyperplane $H \subset \mathbb{C}^p$ for the meromorphic extension of $\int_X |f_1|^{2\lambda_1} \ldots |f_p|^{2\lambda_p}\square$ *is supported by the closed set* $F \subset X$, when H is not a polar hyperplane for the meromorphic extension of $\int_{X\setminus F} |f_1|^{2\lambda_1} \ldots |f_p|^{2\lambda_p}\square$. We shall say that a polar direction is *supported in* F if any polar hyperplane with this direction is supported by F.

Proposition 1.4. *Assume $f_1, \ldots, f_p$ are quasi-homogeneous functions for the weights $w_1, \ldots, w_p$, of degree $a_1, \ldots, a_p$. Then if there exists a polar hyperplane direction supported by the origin for (1.1) in $(\mathbb{N}^*)^p$ it is $(a_1, \ldots, a_p)$ and the corresponding poles are at most simple.*

In particular, for $p = 2$, and if (f_1, f_2) is an icis, all oblique polar lines have direction (a_1, a_2).

Proof. Quasi-homogeneity gives $f_k(t^{w_1}x_1, \ldots, t^{w_m}x_m) = t^{a_k} f_k(x)$, $k = 1, \ldots, p$.

Let $\Omega := \sum_1^m (-1)^{j-1} w_j x_j \, \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_m$ so that $\mathrm{d}\Omega = (\sum w_j)\, \mathrm{d}x$.

From Euler's relation, because $f_k^{\lambda_k}$ is quasi-homogeneous of degree $a_k\lambda_k$:

$$\mathrm{d}f_k^{\lambda_k} \wedge \Omega = a_k\lambda_k f_k^{\lambda_k}\, \mathrm{d}x,$$
$$\mathrm{d}x^\delta \wedge \Omega = \langle w \mid \delta\rangle x^\delta\, \mathrm{d}x, \ \forall \delta \in \mathbb{N}^m.$$

Take $\rho \in \mathcal{C}_c^\infty(\mathbb{C}^m)$; then, with $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^p$ and $\varepsilon \in \mathbb{N}^m$:

$$\mathrm{d}(|f|^{2\lambda} x^\delta \bar{x}^\varepsilon \rho\Omega \wedge \mathrm{d}\bar{x}) =$$
$$= (\langle a \mid \lambda\rangle + \langle w \mid \delta + \mathbf{1}\rangle)|f|^{2\lambda} x^\delta \bar{x}^\varepsilon \rho\, \mathrm{d}x \wedge \mathrm{d}\bar{x} + |f|^{2\lambda} x^\delta \bar{x}^\varepsilon\, \mathrm{d}\rho \wedge \Omega \wedge \mathrm{d}\bar{x}.$$

Using Stokes' formula we get

$$(\langle a \mid \lambda\rangle + \langle w \mid \delta + \mathbf{1}\rangle) \int |f|^{2\lambda} x^\delta \bar{x}^\varepsilon \rho\, \mathrm{d}x \wedge \mathrm{d}\bar{x} = -\int |f|^{2\lambda} x^\delta \bar{x}^\varepsilon\, \mathrm{d}\rho \wedge \Omega \wedge \mathrm{d}\bar{x}.$$

For $\rho = 1$ near 0, $\mathrm{d}\rho = 0$, near 0. Therefore the right-hand side has no poles supported by the origin. Now the conclusion comes from the Taylor expansion at 0 of the test function. □

2. Existence of polar oblique lines

In this section, we consider two holomorphic functions $f, g : Y \to \mathbb{C}$, where Y is an open subset in $\mathbb{C}^m$ and fix a relatively compact open subset X of Y. Without loss of generality, we assume $0 \in X$. We study the possible oblique polar lines of the meromorphic extension of the current valued function

$$(\lambda, \mu) \mapsto \int_X |f|^{2\lambda}|g|^{2\mu}\square. \qquad (2.1)$$

The following elementary lemma is basic.

Lemma 2.1. *Let M be a meromorphic function in $\mathbb{C}^2$ with poles along a family of lines with directions in $\mathbb{N}^2$. For $(\lambda_0, \mu_0) \in \mathbb{C}^2$, assume*

(i) $\{\lambda = \lambda_0\}$ *is a polar line of order* $\leqslant k_0$ *of* M,
(ii) $\{\mu = \mu_0\}$ *is not a polar line of* M,
(iii) $\lambda \mapsto M(\lambda, \mu_0)$ *has a pole of order at least* $k_0 + 1$ *at* λ_0.

Then there exists $(a, b) \in (\mathbb{N}^*)^2$ such that the function M has a pole along the (oblique) line $\{a\lambda + b\mu = a\lambda_0 + b\mu_0\}$.

Proof. If M does not have an oblique polar line through (λ_0, μ_0), then the function $(\lambda, \mu) \mapsto (\lambda - \lambda_0)^{k_0} M(\lambda, \mu)$ is holomorphic near (λ_0, μ_0). Therefore, $\lambda \mapsto M(\lambda, \mu_0)$ has at most a pole of order k_0 at λ_0. Contradiction. □

It turns out that to check the first condition in the above lemma for the function (2.1), a sufficient condition is that the poles of the meromorphic extension of the current valued function

$$\lambda \mapsto \int_X |f|^{2\lambda} \square \tag{2.2}$$

are of order $\leqslant k_0$. Such a simplification does not hold for general meromorphic functions. For example,

$$(\lambda, \mu) \mapsto \frac{\lambda + \mu}{\lambda^2}$$

has a double pole along $\{\lambda = 0\}$ but its restriction to $\{\mu = 0\}$ has only a simple pole at 0.

Proposition 2.2. *If the meromorphic extension of the current valued function* (2.2) *has a pole of order k at $\lambda_0 \in \mathbb{R}_-$,* i.e., *it has a principal part of the form*

$$\frac{T_k}{(\lambda - \lambda_0)^k} + \cdots + \frac{T_1}{\lambda - \lambda_0},$$

at λ_0, then the meromorphic extension of the function (2.1) *has a pole of order*

$$k_0 := \max\{0 \leqslant l \leqslant k \mid \operatorname{supp} T_l \not\subseteq \{g = 0\}\} \tag{2.3}$$

along the line $\{\lambda = \lambda_0\}$.

Proof. As a consequence of the Bernstein identity (see [11]), there exists $N \in \mathbb{N}$ such that the extension of $\int_X |f|^{2\lambda}\phi$ in $\{\Re\lambda > \lambda_0 - 1\}$ can be achieved for $\phi \in \Lambda^{m,m}\mathcal{C}_c^N(X)$. Our hypothesis implies that this function has a pole of order $\leqslant k$ at λ_0. Because $|g|^{2\mu}\phi$ is of class $\mathcal{C}^N$ for $\Re\mu$ large enough, the function

$$\lambda \mapsto \int_X |f|^{2\lambda} |g|^{2\mu} \phi$$

has a meromorphic extension in $\{\Re\lambda > \lambda_0 - 1\}$ with a pole of order $\leqslant k$ at λ_0. We have proved that (2.1) has a pole of order $\leqslant k$ along the line $\{\lambda = \lambda_0\}$.

Near λ_0, the extension of $\int_X |f|^{2\lambda}\phi$ writes

$$\frac{\langle T_k, \phi\rangle}{(\lambda-\lambda_0)^k} + \cdots + \frac{\langle T_1, \phi\rangle}{\lambda-\lambda_0} + \cdots .$$

Hence that of $\int_X |f|^{2\lambda}|g|^{2\mu}\phi$ looks

$$\frac{\langle T_k|g|^{2\mu}, \phi\rangle}{(\lambda-\lambda_0)^k} + \cdots + \frac{\langle T_1|g|^{2\mu}, \phi\rangle}{\lambda-\lambda_0} + \cdots .$$

If $\operatorname{supp} T_k \subseteq \{g=0\}$, then the first term vanishes for $\Re\mu$ large enough, because T_k is of finite order (see the beginning of the proof). So the order of the pole along the line $\{\lambda=\lambda_0\}$ is $\leqslant k_0$.

Take $x_0 \in \operatorname{supp} T_{k_0}$ such that $g(x_0)\neq 0$ and V a neighborhood of x_0 in which g does not vanish. From the definition of the support, there exists $\psi \in \Lambda^{m,m}C_c^\infty(V)$ such that $\langle T_{k_0}, \psi\rangle \neq 0$. With $\phi := \psi|g|^{-2\mu} \in \Lambda^{m,m}C_c^\infty(V)$, we get

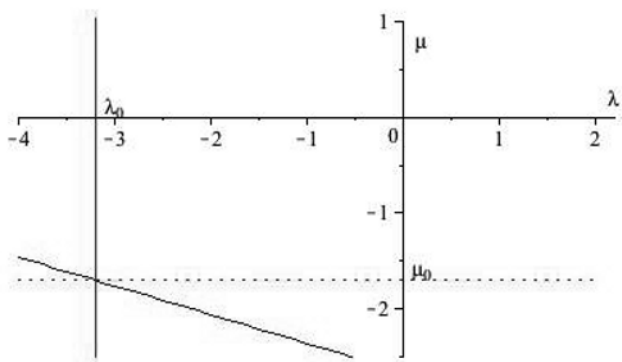
$$\langle T_{k_0}|g|^{2\mu}, \phi\rangle = \langle T_{k_0}, \psi\rangle \neq 0.$$

Therefore, the extension of (2.1) has a pole of order k_0 along the line $\{\lambda=\lambda_0\}$. $\square$

Corollary 2.3. *For $(\lambda_0,\mu_0)\in(\mathbb{R}_-)^2$, assume*

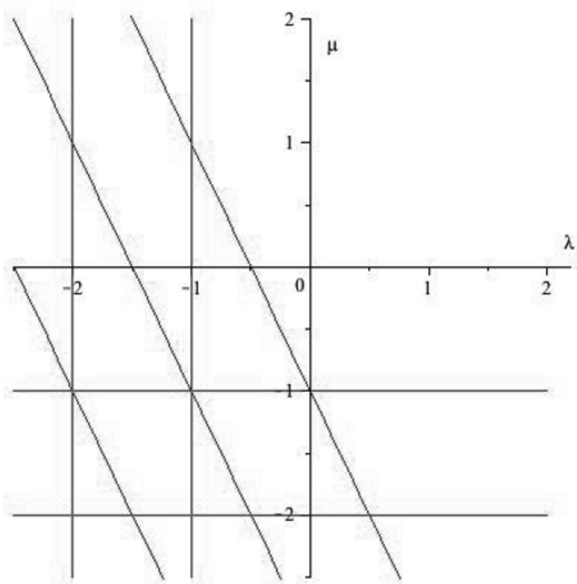
(i) *the extension of the current valued function* (2.2) *has a pole of order k at λ_0,*
(ii) *μ_0 is not an integer translate of a root of the Bernstein polynomial of g,*
(iii) *$\lambda \mapsto \operatorname{Pf}(\mu=\mu_0, \int_X |f|^{2\lambda}|g|^{2\mu}\square)$ has a pole of order $l_0 > k_0$ where k_0 is defined in* (2.3) *at λ_0.*

Then the meromorphic extension of the current valued function (2.1) *has at least l_0-k_0 oblique lines, counted with multiplicities, through (λ_0,μ_0).*

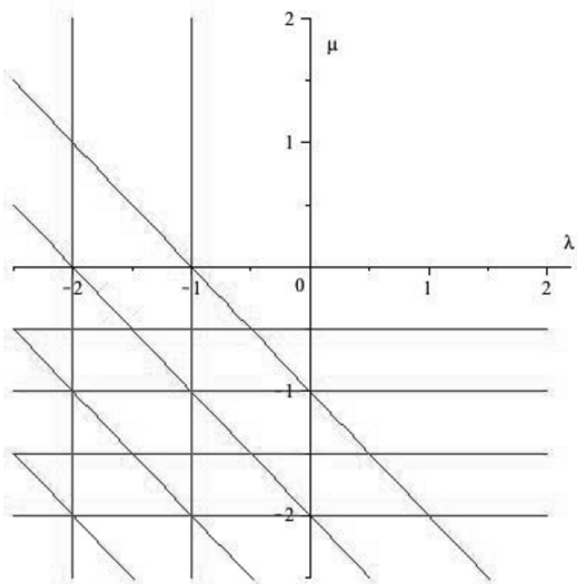


Proof. Use Proposition 2.2 and a version of Lemma 2.1 with multiplicities. $\square$

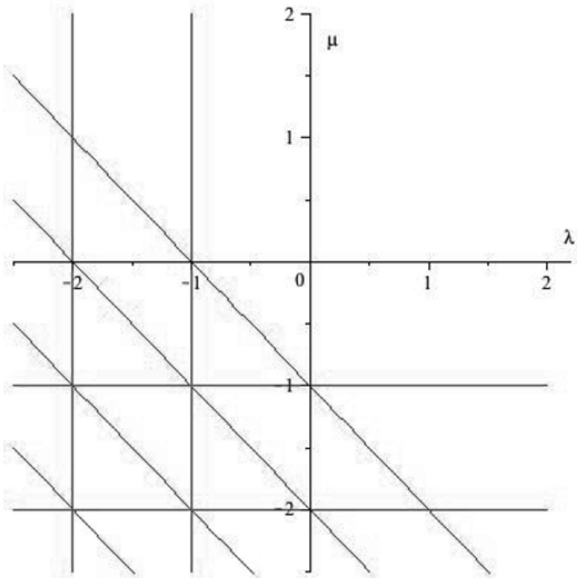
Example 2.4. $m = 3,\ f(x,y,z) = x^2 + y^2 + z^2,\ g(x,y,z) = z\,.$



Example 2.5. $m = 4,\ f(x,y,z) = x^2 + y^2 + z^2 + t^2,\ g(x,y,z,t) = t^2\,.$



Example 2.6. $m = 3,\ f(x,y,z) = x^2 + y^2,\ g(x,y,z) = y^2 + z^2$.



In this last example, Corollary 2.3 does not apply because for $\lambda_0 = -1$ we have $k_0 = l_0$. Existence of an oblique polar line through $(-1,0)$ is obtained by computation of the extension of $\lambda \mapsto \text{Pf}\left(\mu = 1/2, \int_X |f|^{2\lambda}|g|^{2\mu}\square\right)$.

3. Pullback and interaction

In this section, we give by pullback a method to verify condition (iii) of Corollary 2.3 when g is a coordinate. As a matter of fact the function $\lambda \mapsto \int_X |f|^{2\lambda}|g|^{2\mu_0}\square$ is only known by meromorphic extension (via Bernstein identity) when μ_0 is negative; it is in general difficult to exhibit some of its poles.

In $\mathbb{C}^{n+1}$, denote the coordinates by $x_1,\dots,x_n,t$ and take $g(x,t) = t$. We consider therefore only one holomorphic function $f : Y \to \mathbb{C}$, where Y is an open subset in $\mathbb{C}^{n+1}$ and fix a relatively compact open subset X of Y. Let us introduce also the finite map

$$p : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \text{ such that } p(x_1,\dots,x_n,\tau) = (x_1,\dots,x_n,\tau^k) \tag{3.1}$$

for some fixed integer k. Finally, put $\tilde f := f \circ p : \tilde X \to \mathbb{C}$ where $\tilde X := p^{-1}(X)$.

Proposition 3.1. *With the above notations and $\lambda_0 \in \mathbb{R}_-$ suppose*

(a) *the extension of the current valued function* (2.2) *has a pole of order $\leqslant 1$ at λ_0,*

(b) $\lambda \mapsto \int_{\tilde X} |\tilde f|^{2\lambda}\square$ *has a double pole at λ_0.*

Then there exists $l \in [1, k-1]$ such that the extension of the current valued function $\lambda \mapsto \int_X |f|^{2\lambda}|t|^{-2l/k}\square$ has a double pole at λ_0.

Proof. Remark that the support of the polar part of order 2 of $\int_{\tilde{X}} |\tilde{f}|^{2\lambda}\square$ at λ_0 is contained in $\{\tau = 0\}$ because we assume (a) and p is a local isomorphism outside $\{\tau = 0\}$.
After hypothesis (b) there exists $\varphi \in \Lambda^{n+1,n+1}\mathcal{C}_c^\infty(\tilde{X})$ such that

$$A := P_2\big(\lambda = \lambda_0, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\varphi\big) \neq 0.$$

Consider the Taylor expansion of φ along $\{\tau = 0\}$

$$\varphi(x,\tau) = \sum_{j+j' \leqslant N} \tau^j \bar{\tau}^{j'} \varphi_{j,j'}(x) \wedge d\tau \wedge d\bar{\tau} + o(|\tau|^N)$$

where N is larger than the order of the current defined by P_2 on a compact set K containing the support of φ. Therefore

$$A = \sum_{j+j' \leqslant N} P_2\big(\lambda = \lambda_0, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\tau^j\bar{\tau}^{j'}\varphi_{j,j'}(x)\chi(|\tau|^{2k}) \wedge d\tau \wedge d\bar{\tau}\big)$$

where χ has support in K and is equal to 1 near 0. Because A does not vanish there exists $(j,j') \in \mathbb{N}^2$ with $j + j' \leqslant N$ such that

$$A_{j,j'} := P_2\big(\lambda = \lambda_0, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\tau^j\bar{\tau}^{j'}\varphi_{j,j'}(x)\chi(|\tau|^{2k}) \wedge d\tau \wedge d\bar{\tau}\big) \neq 0.$$

The change of variable $\tau \to \exp(2i\pi/k)\tau$ that leaves $\tilde{f}$ invariant, shows that $A_{j,j'} = \exp(2i\pi(j-j')/k)A_{j,j'}$. Hence $A_{j,j'} = 0$ for $j - j' \notin k\mathbb{Z}$. We then get the existence of $(j,j') \in \mathbb{N}^2$ verifying $j' = j + k\nu$ with $\nu \in \mathbb{Z}$ and $A_{j,j'} \neq 0$.

The change of variable $t = \tau^k$ in the computation of $A_{j,j'}$ gives

$$P_2\big(\lambda = -\lambda_0, \int_X |f|^{2\lambda}|t|^{2(j-k+1)/k}\bar{t}^\nu\varphi_{j,j+k\nu}(x)\chi(|t|^2) \wedge dt \wedge d\bar{t}\big) \neq 0.$$

This ends the proof with $-l = j-k+1$ if $\nu \geqslant 0$ and with $-l = j'-k+1$ if $\nu < 0$. Notice that $l < k$ in all cases. Necessarily $l \neq 0$ because from hypothesis (a), we know that the extension of the function (2.2) does not have a double pole at λ_0. □

Theorem 3.2. *For Y open in $\mathbb{C}^{n+1}$ and X relatively compact open subset of Y, let $f : Y \to \mathbb{C}$ be holomorphic and $g(x,t) = t$. Assume (f,g) satisfy properties (1) to (4) of the Introduction. Moreover suppose*

- (a) *$\int |f|^{2\lambda}\square$ has a at most a simple pole at $\lambda_0 - \nu$, $\forall \nu \in \mathbb{N}$;*
- (b) *$e^{2\pi i\lambda_0}$ is a eigenvalue of the monodromy of f acting on the H^{n-1} of the Milnor fiber of f at the generic point of a connected component $S_\imath^*$ of S^*, and there exist a non zero eigenvector such its monodromy around 0 in $S_\imath$ is a primitive k-root of unity, with $k \geqslant 2$;*
- (c) *the degree $d_\imath$ of the covering $t : S_\imath^* \to \mathbb{D}^*$ is prime to k;*
- (d) *$e^{2\pi i\lambda_0}$ is not an eigenvalue of the monodromy of $\tilde{f}$ acting on the H^{n-1} of the Milnor fiber of $\tilde{f}$ at 0, where $\tilde{f}(x,\tau) = f(x_1,\dots,x_n,\tau^k)$.*

Then there exists an oblique polar line of $\int |f|^{2\lambda}|g|^{2\mu}\square$ through $(\lambda_0 - j, -l/k)$, some $j \in \mathbb{N}$, and some $l \in [1, k-1]$.

Remark that condition (b) implies that $\lambda_0 \notin \mathbb{Z}$ because of the result of [1].

Proof. Notice first that $(\lambda,\mu) \mapsto \int |f|^{2\lambda}|g|^{2\mu}\square$ has a simple pole along $\{\lambda = \lambda_0\}$. Indeed the support of the residue current of $\lambda \mapsto \int |f|^{2\lambda}\square$ at λ_0 contains $S_\imath^*$ where t does not vanish and Proposition 2.2 applies.

Denote by z a local coordinate on the normalization of $S_\imath$. The function t has a zero of order $d_\imath$ on this normalization and hence $d_\imath$ is the degree of the cover $S_\imath^* \to \mathbb{D}^*$ induced by t. Without loss of generality, we may suppose $t = z^{d_\imath}$ on the normalization of $S_\imath$.

Lemma 3.3. *Let $k, d \in \mathbb{N}^*$ and put*

$$Y_d := \{(z,\tau) \in \mathbb{D}^2 / \tau^k = z^d\}, Y_d^* := Y_d \setminus \{0\}.$$

If k and d are relatively prime, then the first projection $\mathrm{pr}_{1,d} : Y_d^* \to \mathbb{D}^*$ is a cyclic cover of degree k.

Proof. Let us prove that the cover defined by $\mathrm{pr}_{1,d}$ is isomorphic to the cover defined by $\mathrm{pr}_{1,1}$, that may be taken as definition of a cyclic cover of degree k.

After Bézout's identity, there exist $a, b \in \mathbb{Z}$ such that

$$ak + bd = 1. \tag{3.2}$$

Define $\varphi : Y_d \to \mathbb{C}^2$ by $\varphi(z,\tau) = (z, z^a\tau^b)$. From (3.2), we have $\varphi(Y_d) \subseteq Y_1$ and clearly $\mathrm{pr}_{1,1} \circ \varphi = \mathrm{pr}_{1,d}$.

The map φ is injective because

$$\varphi(z,\tau) = \varphi(z,\tau') \Longrightarrow \tau^k = \tau'^k \text{ and } \tau^b = \tau'^b,$$

hence $\tau = \tau'$, after (3.2). It is also surjective: take $(z,\sigma) \in Y_1^*$; the system

$$\tau^b = \sigma z^{-a}, \ \tau^k = z^d, \text{ when } \sigma^k = z,$$

has a unique solution because the compatibility condition $\sigma^k z^{-ak} = z^{bd}$ is satisfied. $\square$

End of the proof of Theorem 3.2. Take the eigenvector with monodromy $\exp\left(\frac{-2i\pi l}{k}\right)$ with $(l,k) = 1$ given by condition (b). Its pullback by p becomes invariant under the monodromy of τ because of the condition (c) and the lemma given above. After (d), this section does not extend through 0. So we have interaction of strata (see [4]) and a double pole for $\lambda \mapsto \int |\tilde{f}|^{2\lambda}\square$ at $\lambda_0 - j$ with some $j \in \mathbb{N}$. It remains to use Proposition 3.1 and Corollary 2.3. $\square$

4. Interaction of strata revised

The main result of this paragraph is Corollary 4.3 of Theorem 4.2 which guarantees an oblique polar line for the meromorphic extension of

$$\int |f|^{2\lambda}|g|^{2\mu}\square.$$

Notations and hypotheses are those of the introduction. As before, the function g is the last coordinate denoted by t.

We consider the subsheaf for the eigenvalue $\exp(-2i\pi u) \neq 1$ of the local system of vanishing cycles of f along S^*. The main difference with the case of the eigenvalue 1 is that the notion of extendable section at the origin has no a priori meaning. So we shall lift the situation to a suitable branched covering in order to kill this monodromy and to reach the situation studied in [4].

Using realization of vanishing cycles via holomorphic differential forms, we give a precise meaning downstairs to the notion of extendable section at the origin in term of the sheaf $\underline{H}^n_{[S]}(\mathcal{O}_X)$. This will be achieved in Theorems 4.1 and 4.2 which explain the extendable case and the non extendable case for a given suitable multivalued section on S^*.

We suppose that the eigenvalue $\exp(-2i\pi u)$ of the monodromy of f is simple at each point of S^*. Therefore, this eigenvalue is also simple for the monodromy acting on the group H^{n-1} of the Milnor fibre of f at 0. In order to compute the constructible sheaf $H^{n-1}(u)$ on S we may use the complex $(\Omega_X[f^{-1}], \delta_u)$, that is the complex of meromorphic forms with poles in $f^{-1}(0)$ equipped with the differential $\delta_u := d - u\frac{df}{f}\wedge$ along S. This corresponds to the case $k_0 = 1$ in [4].

We use the isomorphisms

$$r^{n-1} : h^{n-1} \to H^{n-1}(u) \text{ over } S \quad \text{and} \quad \tau_1 : h^{n-1} \to h^n \text{ over } S^*, \tag{4.1}$$

where h^{n-1} [resp. h^n] denotes the $(n-1)$th [resp. nth] cohomology sheaf of the complex $(\Omega_X[f^{-1}], \delta_u)$.

In order to look at the eigenspace for the eigenvalue $\exp(-2i\pi l/k)$ of the monodromy Θ of the local system $H^{n-1}(u)$ on S^*, it will be convenient to consider the complex of sheaves

$$\Gamma_l := \big(\Omega^\bullet[f^{-1}, t^{-1}], \delta_u - \frac{l}{k}\frac{dt}{t}\wedge\big)$$

which is locally isomorphic along S^* to $(\Omega_X[f^{-1}], \delta_u)$ via the choice of a local branch of $t^{l/k}$ and the morphism of complexes $(\Omega_X[f^{-1}], \delta_u) \to \Gamma_l$ which is given by $\omega \mapsto t^{-l/k}.\omega$ and satisfies

$$\delta_u(t^{-l/k}\omega) - \frac{l}{k}\frac{dt}{t} \wedge t^{-l/k}\omega = t^{-l/k}\delta_u(\omega).$$

But notice that this complex Γ_l is also defined near the origin. Of course, a global section $\sigma \in H^0(S^*, h^{n-1}(\Gamma_l))$ gives, via the above local isomorphism, a multivalued global section on S^* of the local system $H^{n-1}(u) \simeq h^{n-1}$ with monodromy $\exp(-2i\pi l/k)$ (as multivalued section).

So a global meromorphic differential $(n-1)$-form ω with poles in $\{f.t = 0\}$ such that $d\omega = u\frac{df}{f} \wedge \omega + \frac{l}{k}\frac{dt}{t} \wedge \omega$ defines such a σ, and an element in $\mathcal{H}_0$ with monodromy $\exp(-2i\pi l/k)$.

We shall use also the morphism of complexes of degree +1

$$\check{\tau}_1 : \Gamma_l \to \Gamma_l$$

given by $\check{\tau}_1(\sigma) = \frac{df}{f} \wedge \sigma$. It is an easy consequence of [4] that in our situation $\check{\tau}_1$ induces an isomorphism $\check{\tau}_1 : h^{n-1}(\Gamma_l) \to h^n(\Gamma_l)$ on S^*, because we have assumed that the eigenvalue $\exp(-2i\pi u)$ for the monodromy of f is simple along S^*.

Our first objective is to build for each $j \in \mathbb{N}$ a morphism of sheaves on S^*

$$r_j : h^{n-1}(\Gamma_l) \to \underline{H}^n_{[S]}(\mathcal{O}_X), \tag{4.2}$$

via the meromorphic extension of $\int_X |f|^{2\lambda}|t|^{2\mu}\square$. Here $\underline{H}^n_{[S]}(\mathcal{O}_X)$ denotes the subsheaf of the moderate cohomology with support S of the sheaf $\underline{H}^n_S(\mathcal{O}_X)$. It is given by the nth cohomology sheaf of the Dolbeault-Grothendieck complex with support S:

$$\underline{H}^n_{[S]}(\mathcal{O}_X) \simeq \mathcal{H}^n(\underline{H}^0_S(\mathcal{D}b_X^{0,\bullet}), \mathrm{d}'').$$

Let w be a $(n-1)$-meromorphic form with poles in $\{f=0\}$, satisfying $\delta_u(w) = \frac{l}{k}\frac{\mathrm{d}t}{t} \wedge w$ on an open neighbourhood $U \subset X \setminus \{t=0\}$ of a point in S^*. Put for $j \in \mathbb{N}$:

$$\overline{r_j(w)} := \mathrm{Res}\left(\lambda = -u, \mathrm{Pf}\left(\mu = -l/k, \int_U |f|^{2\lambda}|t|^{2\mu}\bar{f}^{-j}\frac{\mathrm{d}f}{f} \wedge w \wedge \square\right)\right).$$

Here $\mathrm{Res}\big(\lambda = -u, \dots\big)$ denotes the residue at $\lambda = -u$ and $Pf(\mu = -l/k, \dots)$ is the constant term in the Laurent expansion at $\mu = -l/k$.

These formulae define d'-closed currents of type $(n,0)$ with support in $S^* \cap U$.

Indeed it is easy to check that the following formula holds in the sense of currents on U:

$$\mathrm{d}'\big[\mathrm{Pf}\left(\lambda = -u, \mathrm{Pf}\left(\mu = -l/k, \int_U |f|^{2\lambda}|t|^{2\mu}\bar{f}^{-j} w \wedge \square\right)\right)\big]$$
$$= \mathrm{Res}\left(\lambda = -u, \mathrm{Pf}\left(\mu = -l/k, \int_U |f|^{2\lambda}|t|^{2\mu}\bar{f}^{-j}\frac{\mathrm{d}f}{f} \wedge w \wedge \square\right)\right).$$

On the other hand, if $w = \delta_u(v) - \frac{l}{k}\frac{\mathrm{d}t}{t} \wedge v$ for $v \in \Gamma(U, \Omega^{n-2}[f^{-1}])$, then

$$\mathrm{d}'\big[\mathrm{Res}\left(\lambda = -u, \mathrm{Pf}\left(\mu = -l/k, \int_U |f|^{2\lambda}|t^{2\mu}\bar{f}^{-j}\frac{\mathrm{d}f}{f} \wedge v \wedge \square\right)\right)\big]$$
$$= \mathrm{Res}\left(\lambda = -u, \mathrm{Pf}\left(\mu = -l/k, \int_U |f|^{2\lambda}|t|^{2\mu}\bar{f}^{-j}\frac{\mathrm{d}f}{f} \wedge w \wedge \square\right)\right)$$

because the meromorphic extension of $\int_X |f|^{2\lambda}\square$ has no double poles at $\lambda \in -u - \mathbb{N}$ along S^*, since $\exp(-2i\pi u)$ is a simple eigenvalue of the monodromy of f along S^*. It follows that the morphism of sheaves (4.2) is well defined on S^*.

By direct computation we show the following equality between sections on S^* of the sheaf $\underline{H}^n_{[S]}(\Omega^1_X)$:

$$\mathrm{d}' r_j(w) = -(u+j)\,\mathrm{d}f \wedge r_{j+1}(w)$$

where $\mathrm{d}' : \underline{H}^n_{[S]}(\mathcal{O}_X) \to \underline{H}^n_{[S]}(\Omega^1_X)$ is the morphism induced by the de Rham differential $\mathrm{d} : \mathcal{O}_X \to \Omega^1_X$.

Because $\underline{H}^n_{[S]}(\mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X$-modules, it is possible to define the product $g.r_j$ for g holomorphic near a point of S^* and the usual rule holds

$$\mathrm{d}'(g.r_j) = \mathrm{d}g \wedge r_j + g.\, \mathrm{d}'\rho_j.$$

Now we shall define, for each irreducible component $S_\imath$ of S such that the local system $H^{n-1}(u)^\imath$ has $\exp(-2i\pi l/k)$ as eigenvalue for its monodromy $\Theta^\imath$, linear maps

$$\rho_j^\imath : \mathrm{Ker}\left(\Theta^\imath - \exp(-2i\pi l/k)\right) \to H^0(S_\imath^*, \underline{H}^n_{[S]}(\mathcal{O}_X))$$

as follows:
Let $s_\imath \in S_\imath^*$ be a base point and let $\gamma \in H^{n-1}(u)_{s_\imath}$ be such that $\Theta^\imath(\gamma) = \exp(2i\pi l/k).\gamma$. Denote by $\sigma(\gamma)$ the multivalued section of the local system $H^{n-1}(u)^\imath$ on $S_\imath^*$ defined by γ. Near each point of $s \in S_\imath^*$ we can induce σ by a meromorphic $(n-1)$-form w_0 which is δ_u-closed. Choose a local branch of $t^{1/k}$ near the point s and put $w := t^{l/k} w_0$. Then it is easy to check that we define in this way a global section $\Sigma(\gamma)$ on $S_\imath^*$ of the sheaf $h^{n-1}(\Gamma_l)$ which is independent of our choices. Now set

$$\rho_j^\imath(\gamma) := r_j(\Sigma(\gamma)).$$

Like in Section 3, define $\tilde{f} : \tilde{X} \to \mathbb{C}$ by $\tilde{f} := f \circ p$ with p of (3.1). The singular locus $\tilde{S}$ of $\tilde{f}$ is again a curve, but it may have components contained in $\{\tau = 0\}$ (see for instance Example 5.1). Let $\tilde{S}^* := \tau^{-1}(\mathbb{D}^*)$ (so in $\tilde{S}$ we forget about the components that are in $\tau^{-1}(0)$) and define the local system $\tilde{\mathcal{H}}$ on $\mathbb{D}^*$ as $\tau_*(\tilde{H}^{n-1}(u)|_{\tilde{S}^*})$. Denote its fiber $\tilde{\mathcal{H}}_0$ at some τ_0 with $\tau_0^k = t_0$ and the monodromy $\tilde{\Theta}_0$ of $\tilde{\mathcal{H}}_0$.

We have

$$\tilde{\Theta}_0 = (\pi_*)^{-1} \circ \Theta_0 \circ \pi_*, \text{ where } \pi(\tau) := \tau^k,$$

and $\pi_* : \tilde{\mathcal{H}}_0 \to \mathcal{H}_0$ is the isomorphism induced by π.

Choose now the base points $s_\imath$ of the connected components $S_\imath^*$ of S^* in $\{t = t_0\}$ where t_0 is the base point of $\mathbb{D}^*$. Moreover choose the base point $\tau_0 \in \mathbb{C}^*$ such that $\tau_0^k = t_0$.

In order to use the results of [4], we need to guarantee that for the component $S_\imath^*$ of S^*, the map $p^{-1}(S_\imath^*) \to S_\imath^*$ is the cyclic cover of degree k.
Fix a base point $\tilde{s}_\imath \in p^{-1}(S_\imath^*)$ such that $p(\tilde{s}_\imath) = s_\imath$. The local system $\tilde{H}^{n-1}(u)$ on the component $\tilde{S}_\imath^*$ of $\tilde{S}^*$ containing $\tilde{s}_\imath$ is given by $\tilde{H}^{n-1}(u)_{\tilde{s}_\imath}$ which is isomorphic to $H^{n-1}(u)_{s_\imath}$, and the monodromy automorphism $\tilde{\Theta}^\imath$. In case $p : \tilde{S}_\imath^* \to S_\imath^*$ is the cyclic cover of degree k, we have $\tilde{\Theta}^\imath = (\Theta^\imath)^k$.

After Lemma 3.3, this equality is true if k is prime to the degree $d_\imath$ of the covering $t : S_\imath^* \to \mathbb{D}^*$.

Let $\tilde{\gamma}$ be the element in $(\tilde{H}^{n-1}(u)^\imath)_{\tilde{s}_\imath}$ whose image by p is γ. Let $\sigma(\tilde{\gamma})$ the multivalued section of the local system $\tilde{H}^{n-1}(u)^\imath$ on $\tilde{S}_\imath^*$ given by $\tilde{\gamma}$ on $\tilde{S}_\imath^*$. By construction, if $(k, d_\imath) = 1$, we get $\tilde{\Theta}^\imath\tilde{\gamma} = \tilde{\gamma}$. Therefore $\sigma(\tilde{\gamma})$ is in fact a global (singlevalued) section of the local system $\tilde{H}^{n-1}(u)^\imath$ over $\tilde{S}^*_\imath$.

Theorem 4.1. *Notations and hypotheses are those introduced above. Take γ in $H^{n-1}(u)_{s_\imath}$ such that $\Theta^\imath(\gamma) = \exp(-2i\pi l/k)\gamma$ where $\Theta^\imath$ is the monodromy of $H^{n-1}(u)_{s_\imath}$ and l is an integer prime to k, between 1 and $k-1$.*
Assume that k is relatively prime to the degree of the cover $t|_{S^*_\imath}$ of $\mathbb{D}^*$.
If the section $\sigma(\tilde{\gamma})$ of $\tilde{H}^{n-1}(u)^\imath$ on $\tilde{S}^*_\imath$ defined by γ is the restriction to $\tilde{S}^*_\imath$ of a global section W on $\tilde{S}$ of the constructible sheaf $\tilde{H}^{n-1}(u)$ then there exists $\omega \in \Gamma(X, \Omega^{n-1}_X)$ such that the following properties are satisfied:

(1) $\mathrm{d}\omega = (m+u)\dfrac{\mathrm{d}f}{f} \wedge \omega + \dfrac{l}{k}\dfrac{\mathrm{d}t}{t} \wedge \omega$, *for some $m \in \mathbb{N}$;*

(2) *The $(n-1)$-meromorphic δ_u-closed form $t^{-l/k}\omega/f^m$ induces a section on S of the sheaf $h^{n-1}(\Gamma_l)$ whose restriction to $S^*_\imath$ is given by $\Sigma(\gamma)$;*

(3) *the current*

$$T_j := \mathrm{Res}\left(\lambda = -m-u, Pf\big(\mu = -l/k, \int_X |f|^{2\lambda}\bar{f}^{m-j}|t|^{2\mu}\frac{\mathrm{d}f}{f} \wedge \omega \wedge \square\big)\right)$$

satisfies $\mathrm{d}'T_j = \mathrm{d}'K_j$ for some current K_j supported in the origin and $T_j - K_j$ is a $(n,0)$-current supported in S whose conjugate induces a global section on S of the sheaf $\underline{H}^n_{[S]}(\mathcal{O}_X)$ which is equal to $r_j(\gamma)$ on $S^*_\imath$.

Proof. After [2] and [4], there exist an integer $m \geqslant 0$ and a $(n-1)$-holomorphic form $\tilde{\omega}$ on X verifying the following properties:

(i) $\mathrm{d}\tilde{\omega} = (m+u)\dfrac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega}$;

(ii) along $\tilde{S}$ the meromorphic δ_u-closed form $\dfrac{\tilde{\omega}}{\tilde{f}^m}$ induces the section W;

(iii) the current

$$\tilde{T}_j := \mathrm{Res}\left(\lambda = -m-u, \int_{\tilde{X}} |\tilde{f}|^{2\lambda}\bar{\tilde{f}}^{m-j}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega} \wedge \ \square\right)$$

satisfies $\mathrm{d}'\tilde{T}_j = \mathrm{d}'\tilde{K}_j$ for some current $\tilde{K}_j$ supported in the origin and $\tilde{T}_j - \tilde{K}_j$ is a $(n,0)$-current supported in $\tilde{S}$ whose conjugate induces the element $r_j(W)$ of $H^n_{\tilde{S}}(\tilde{X}, \mathcal{O}_{\tilde{X}})$.

On $\tilde{X}$ we have an action of the group $\mathfrak{G}_k$ of kth roots of unity that is given by $Z.(x,\tau) := (x, \zeta\tau)$ where $\zeta := \exp(2i\pi/k)$. Then X identifies to the complex smooth quotient of $\tilde{X}$ by this action. In particular every $\mathfrak{G}_k$-invariant holomorphic form on $\tilde{X}$ is the pullback of a holomorphic form on X. For the holomorphic form $\tilde{\omega} \in \Gamma(\tilde{X}, \Omega^{n-1}_{\tilde{X}})$ above, we may write

$$\tilde{\omega} = \sum_{l=0}^{k-1} \tilde{\omega}_\ell, \qquad \text{with} \quad Z^*\tilde{\omega}_\ell = \zeta^\ell\tilde{\omega}_\ell.$$

Indeed, $\tilde{\omega}_\ell = \frac{1}{k}\sum_{j,\ell=0}^{k-1}\zeta^{-j\ell}(Z^j)^*\tilde{\omega}$ does the job. Because $\tau^{k-\ell}\tilde{\omega}_\ell$ is $\mathfrak{G}_k$-invariant, there exist holomorphic forms $\omega_0,\dots,\omega_{k-1}$ on X such that

$$\tilde{\omega} = \sum_{\ell=0}^{k-1}\tau^{\ell-k}p^*\omega_\ell. \tag{4.3}$$

Put $\omega := \omega_{k-l}$. Because property (i) above is Z-invariant, each ω_ℓ verifies it and hence ω, whose pullback by p is $\tau^{k-l}\tilde{\omega}_{k-l}$, will satisfy the first condition of the theorem, after the injectivity of p^*.

The action of Z on $\tilde{\gamma}$ is $Z\tilde{\gamma} = \zeta^{-l}\tilde{\gamma}$; therefore $\tilde{\omega}_{k-l}$ verifies (ii) above and hence for $\ell \neq k-l$, the form $\tilde{\omega}_\ell$ induces 0 in $\tilde{H}^{n-1}(u)$ along $\tilde{S}$.

Let us prove property (3) of the theorem. When $\tilde{\omega}$ is replaced by $\tilde{\omega}_{k-l}$ in the definition of $\tilde{T}_j$, the section it defines on $\tilde{S}$ does not change. On the other hand, the action of Z on this section is given by multiplication by ζ^{-l}. Because this section extends through 0, the same is true for $\tau^{k-l}\tilde{T}_j$ whose conjugate will define a $\mathfrak{G}_k$-invariant section of $\underline{H}^n_{[S]}(\mathcal{O}_{\tilde{X}})$ extendable through 0. Condition (3) follows from the isomorphism of the subsheaf of $\mathfrak{G}_k$-invariant sections of $\underline{H}^n_{[\tilde{S}]}(\mathcal{O}_{\tilde{X}})$ and $\underline{H}^n_{[S]}(\mathcal{O}_X)$. □

Our next result treats the case where there is a section $\tilde{W}$ of $\tilde{H}^{n-1}(u)$ on $\tilde{S}^*$ which is *not extendable at the origin* and induces $\tilde{\gamma}$ on $\tilde{S}^*_\imath$. Remark that there always exists a global section on $\tilde{S}^*$ inducing $\tilde{\gamma}$ on $\tilde{S}^*_\imath$: just put 0 on the branches $\tilde{S}^*_{\imath'}$ for each $\imath' \neq \imath$.

The next theorem shows that in the case where our section on $\tilde{S}^*$ is not extendable at the origin, we obtain an oblique polar line for $\int_X |f|^{2\lambda}|g|^{2\mu}\square$.

Remark that in any case we may apply the previous theorem or the next one. When S^* is not connected, it is possible that both apply, because it may exist at the same time a global section on $\tilde{S}^*$ of the sheaf $\tilde{H}^{n-1}(u)$ inducing $\tilde{\gamma}$ on $\tilde{S}^*_\imath$ which is extendable at the origin, and another one which is not extendable at the origin.

Theorem 4.2. *Under the hypotheses of Theorem 4.1, assume that we have a global section $\tilde{W}$ on $\tilde{S}^*$ of the local system $\tilde{H}^{n-1}(u)$ inducing $\tilde{\gamma}$ on $\tilde{S}^*_\imath$ which is not extendable at the origin. Then there exists $\Omega \in \Gamma(X, \Omega^n_X)$ and $l' \in [1,k]$ with the following properties:*

(1) $\mathrm{d}\Omega = (m+u)\dfrac{\mathrm{d}f}{f}\wedge\Omega + \dfrac{l'}{k}\dfrac{\mathrm{d}t}{t}\wedge\Omega$;

(2) *along S^* the n-meromorphic $(\delta_u - \frac{l'}{k}\frac{\mathrm{d}t}{t}\wedge)$-closed form Ω/f^m induces $\check{\tau}_1(\sigma) = \frac{\mathrm{d}f}{f}\wedge\sigma$ in the sheaf $h^n(\Gamma_{l'})$, for some global section σ on S^* of the sheaf $h^{n-1}(\Gamma_{l'})$;*

(3) *the current on X of type $(n+1,0)$ with support $\{0\}$:*

$$P_2\Big(\lambda = -m-u, Pf\big(\mu = -(k-l')/k, \int_X |f|^{2\lambda}\bar{f}^{m-j}|t|^{2\mu}\frac{\mathrm{d}f}{f}\wedge\Omega\wedge\square\big)\Big)$$

defines a non zero class in $H^{n+1}_{[0]}(X,\mathcal{O}_X)$ for j large enough in $\mathbb{N}$.

As a direct consequence, with the help of Corollary 2.3, we obtain:

Corollary 4.3. *Under the hypotheses of Theorem* 4.2*, we get an oblique polar line of* $\int_X |f|^{2\lambda}|g|^{2\mu}\square$ *through* $(-u-j, -l'/k)$ *for* $j \gg 1$ *and some* $l' \in [1, k-1]$*, provided* $\int_X |f|^{2\lambda}\square$ *has only simple poles at* $-u-q$*, for all* $q \in \mathbb{N}$*.*

Proof. As our assumption implies that $H^1_0(\tilde{S}, \tilde{H}^{n-1}(u)) \neq 0$, Proposition 10 and Theorem 13 of [4] imply the existence of $\tilde{\Omega} \in \Gamma(\tilde{X}, \Omega^n_{\tilde{X}})$ verifying

(i) $\mathrm{d}\tilde{\Omega} = (m+u)\dfrac{\mathrm{d}\tilde{f}}{f} \wedge \tilde{\Omega}$, for some $m \in \mathbb{N}$;

(ii) $\tilde{ob}_1([\tilde{\Omega}]) \neq 0$, that is $\tilde{\Omega}/\tilde{f}^m$ induces, via the isomorphisms (4.1), an element in $H^0(\tilde{S}^*, \tilde{H}^{n-1}(u))$ which is not extendable at the origin;

(iii) $Z\tilde{\Omega} = \zeta^{-l'}\Omega$, for some $l' \in [1, k]$ and $\zeta := \exp(-2i\pi/k)$.

Define then $\gamma' \in \mathcal{H}_0$ by the following condition: $(\pi_*)^{-1}\gamma'$ is the value at τ_0 of $\widetilde{\mathrm{ob}}_1([\tilde{\Omega}])$. After condition (iii) we have $\Theta_0(\gamma') = \zeta^{-l'}\gamma'$.

As we did in (4.3), we may write

$$\tilde{\Omega} = \sum_{\ell=0}^{k-1} \tau^{\ell-k} p^*\Omega_\ell.$$

Put $\Omega := \Omega_{k-l'}$. Because $p^*\Omega = \tau^{k-l'}\tilde{\Omega}_{k-l'}$ and $\tilde{\Omega}_\ell$ satisfies $\mathrm{d}\tilde{\Omega}_\ell = (m+u)\frac{\mathrm{d}\tilde{f}}{f} \wedge \tilde{\Omega}_\ell$ for any ℓ, property (1) of the theorem is satisfied thanks to injectivity of p^*.

Relation (iii) implies that $\tilde{\Omega}_{k-l'}$ induces $\tilde{\gamma}' := (\pi_*)^{-1}\gamma'$ and $\tilde{\Omega}_\ell$ induces 0 for $\ell \neq k-l'$. Hence condition (2) of the theorem is satisfied.

In order to check condition (3), observe that the image of $r_j(\tilde{\gamma}')$ in $H^{n+1}_{[0]}(\tilde{X}, \Omega^{n+1}_{\tilde{X}})$ is equal to the conjugate of

$$\begin{aligned} \mathrm{d}' \operatorname{Res}\big(\lambda = -m-u, \int_{\tilde{X}} |\tilde{f}|^{2\lambda} \bar{\tilde{f}}^{-j} \tilde{\Omega}_{k-l'} \wedge \square\big) \\ = P_2\big(\lambda = -m-u, \int_{\tilde{X}} |\tilde{f}|^{2\lambda} \bar{\tilde{f}}^{-j} \frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\Omega}_{k-l'} \wedge \square\big). \end{aligned}$$

After [4], this current is an analytic nonzero functional supported in the origin in $\tilde{X}$. There exists therefore $\tilde{w} \in \Gamma(\tilde{X}, \tilde{\Omega}^{n+1}_{\tilde{X}})$ such that

$$P_2(\lambda = -m-u, \int_{\tilde{X}} |\tilde{f}|^{2\lambda} \bar{\tilde{f}}^{-j} \frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\Omega}_{k-l'} \wedge \chi\bar{\tilde{w}}) \neq 0,$$

for any cutoff χ equal to 1 near 0. The change of variable $\tau \mapsto \zeta\tau$ shows that $\tilde{w}$ may be replaced by its component $\tilde{w}_{k-l'}$ in the above relation. With $w \in \Gamma(X, \Omega^{n+1}_X)$ such that $p^*w = \tau^{k-l'}\tilde{w}_{k-l'}$ we get

$$P_2\big(\lambda = -m-u, \int_X |f|^{2\lambda} \bar{f}^{-j} |t|^{-2(k-l')/k} \frac{\mathrm{d}f}{f} \wedge \Omega_{k-l'} \wedge \chi\bar{w}\big) \neq 0. \qquad \square$$

Remark 4.4. The case $l' = k$ is excluded because we assume that $\int_X |f|^{2\lambda}\square$ has only simple poles at $-u-q$ for all $q \in \mathbb{N}$. Indeed, if $l' = k$, the last formula of the proof above contradicts our assumption.

5. Examples

Example 5.1. $n = 2$, $f(x,y,t) = tx^2 - y^3$. We shall show that both Theorems 4.1 and 4.2 may be applied in this example. Thanks to Corollary 4.3 we obtain that the extension of $\int_X |f|^{2\lambda}|t|^{2\mu}\square$ presents an oblique polar line of direction $(3,1)$ through $(-5/6-j, -1/2)$, for $j \gg 1$. In fact it follows from general facts that $j = 2$ is large enough because here X is a neighborhood of 0 in $\mathbb{C}^3$.

Proof. We verify directly that the standard generator of $H^1(5/6)$ (which is a local system of rank 1) on $S^* := \{x = y = 0\} \cap \{t \neq 0\}$ has monodromy $-1 = \exp(2i\pi.1/2)$. We take therefore $k = 2$ and we have $\tilde{f}(x,y,\tau) := \tau^2 x^2 - y^3$.

Put

$$\tilde{S}^* = \tilde{S}_1^* \cup \tilde{S}_2^* \text{ with } \tilde{S}_1^* := \{x = y = 0\} \cap \{\tau \neq 0\},\ \tilde{S}_2^* := \{\tau = y = 0\} \cap \{x \neq 0\}.$$

The form $\tilde{\omega} := 3x\tau\, dy - 2y\, \mathrm{d}(x\tau)$ verifies

$$\mathrm{d}\tilde{\omega} = \frac{5}{6}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega} \tag{5.1}$$

and $\tilde{\omega}$ induces a nonzero element in the H^1 of the Milnor fibre of $\tilde{f}$ at 0 because it induces on $\tilde{S}_1^*$ the pullback of the multivalued section of $H^1(5/6)$ we started with. It follows that the form ω of Theorem 4.1 is

$$\omega = 3xt\, dy - 2yt\, dx - xy\, dt.$$

It verifies $p^*\omega = \tau\tilde{\omega}$ and hence

$$\mathrm{d}\omega = \frac{5}{6}\frac{\mathrm{d}f}{f} \wedge \omega + \frac{1}{2}\frac{dt}{t} \wedge \omega.$$

Theorem 4.2 may be used to see the existence of an oblique polar line as follows. Construct a section on $\tilde{S}^*$ of $\tilde{H}^1(5/6)$ by setting 0 on $\tilde{S}_2^*$ and the restriction of $\tilde{\omega}$ to $\tilde{S}_1^*$. The point is now to prove that this section is not extendable at the origin. Using the quasi-homogeneity of $\tilde{f}$ it is not difficult to prove by a direct computation that a holomorphic 1-form near the origin satisfying (5.1) (it is enough to look at quasi-homogeneous forms of weight 10) is proportional to $\tilde{\omega}$. As $\tilde{\omega}$ is invariant by the automorphism $(\tau, x, y) \to (x, \tau, y)$ which leaves $\tilde{f}$ invariant and exchanges $\tilde{S}_1$ and $\tilde{S}_2$, we see that $\tilde{\omega}$ does not induce the zero section of $\tilde{H}^1(5/6)$ on $\tilde{S}_2^*$. This proves the non extendability of the section on $\tilde{S}^*$ defined above.

Another way to see the interaction of strata for $\tilde{f}$ for the eigenvalue $\exp(-2i\pi 5/6)$ consists in looking at the holomorphic 2-form $\tilde{\Omega} := \frac{d\tau}{\tau} \wedge \tilde{\omega} = d\tau \wedge (3x\, dy - 2y\, dx)$ that verifies $\mathrm{d}\tilde{\Omega} = \frac{5}{6}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\Omega}$.

Along $\tilde{S}_1^*$ we have

$$\tilde{\Omega} = \mathrm{d}(\tilde{\omega}\log\tau) - \frac{5}{6}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega}\log\tau = \delta_{5/6}(\tilde{\omega}\log\tau),$$

after (5.1). Hence the obstruction to write $\tilde{\Omega}$ near the origin as $\delta_{5/6}(\alpha)$, where α is an holomorphic 1-form near 0 is not zero. More precisely, using [4] we have $\theta(\tilde{\Omega}) \neq 0$ in $H^1(S_1^*, \tilde{H}^1(5/6))$, and also in

$$H^1(S^*, \tilde{H}^1(5/6)) \simeq H^1(S_1^*, \tilde{H}^1(5/6)) \oplus H^1(S_1^*, \tilde{H}^1(5/6)).$$

This implies that the interaction of strata occurs.
But this implies also from [4] that $H^1_{\{0\}}(\tilde{S}, \tilde{H}^1(5/6)) \neq 0$, from which we can conclude that $\dim H^0(\tilde{S}, \tilde{H}^1(5/6))$ is strictly less than the dimension of the space $H^0(\tilde{S}^*, \tilde{H}^1(5/6))$ which is equal to 2. This allows to avoid the direct computation on the (quasi-homogeneous) holomorphic forms to evaluate the dimension of the vector space $H^0(\tilde{S}, \tilde{H}^1(5/6))$ via the complex with differential $\delta_{5/6}$.

Notice also that the meromorphic extension of $\int_X |f|^{2\lambda}\square$ does not have a double pole at $-5/6 - j$, for all $j \in \mathbb{N}$, because interaction of strata is not present for f at 0 for the eigenvalue $\exp(-2i\pi 5/6)$: the monodromies for H^1 and H^2 of the Milnor fibre of f at 0 do not have the eigenvalue $\exp(-2i\pi 5/6)$ because they are of order 3 thanks to homogeneity. $\square$

Example 5.2. $n = 2$, $f(x, y, t) = x^4 + y^4 + tx^2y$. The extension of $\int_X |f|^{2\lambda}|t|^{2\mu}\square$ presents an oblique polar line of direction $(4, 1)$ through $(-5/8, -1/2)$.

Proof. The Jacobian ideal of f relative to t, denoted by $J_/(f)$, is generated by

$$\frac{\partial f}{\partial x} = 4x^3 + 2txy \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y^3 + tx^2.$$

We have

$$t\frac{\partial f}{\partial x} - 4x\frac{\partial f}{\partial y} = 2(t^2 - 8y^2)xy. \tag{5.2}$$

Put $\delta := t^2 - 8y^2$ and notice that for $t \neq 0$ the function δ is invertible at $(t, 0, 0)$. We use notations and results of [5]. Recall that $\mathbb{E} := \Omega^2_/ / \mathrm{d}_/ f \wedge \mathrm{d}_/\mathcal{O}$ is equipped with two operations a and b defined by $a\xi = \xi f$, $b(\mathrm{d}_/\xi) := \mathrm{d}_/ f \wedge \xi$ and a t-connection $b^{-1}.\nabla : \mathbb{P} \to \mathbb{E}$ that commutes to a and b where

1. $\nabla : \mathbb{E} \to \mathbb{E}$ is given by $\nabla(\mathrm{d}_/\xi) := \mathrm{d}_/ f \wedge \frac{\partial \xi}{\partial t} - \frac{\partial f}{\partial t} d\xi$,
2. $\mathbb{P} := \{\alpha \in \mathbb{E} \mid \nabla(\alpha) \in b\mathbb{E}\}$.

Relation (5.2) gives

$$2xy\delta = \mathrm{d}_/ f \wedge (t\,dy + 4x\,dx) = \mathrm{d}_/ f \wedge \mathrm{d}_/(ty + 2x^2); \tag{5.3}$$

hence $xy\delta = 0 \in \mathbb{E}$, and $xy \in J_/(f)$ for $t \neq 0$. As a consequence, for $t_0 \neq 0$ fixed, x^3 and y^4 belong to $J(f_{t_0})$. Therefore the (a, b)-module $\mathbb{E}_{t_0} := \mathbb{E}/(t - t_0)\mathbb{E}$ has rank 5 over $\mathbb{C}[[b]]$. The elements $1, x, y, x^2, y^2$ form a basis of this module.

We compute now the structure of (a, b)-module of $\mathbb{E}$ over the set $\{t \neq 0\}$, *i.e.*, compute the action of a on the basis. Let us start with

$$a(y^2) = x^4y^2 + y^6 + tx^2y^3.$$

Relation (5.2) yields

$$2x^2y\delta = \mathrm{d}_/ f \wedge (tx\,dy + 4x^2\,dx) = t.b(1) \tag{5.4}$$

and also

$$x^2y = b\left(\mathrm{d}\left(\frac{tx\,dy + 4x^2\,dx}{2\delta}\right)\right) = \frac{1}{2t}b(1) + \frac{4}{t^3}b(y^2) + b^2.\mathbb{E}. \tag{5.5}$$

From

$$b(1) = \mathrm{d}_/ f \wedge (x\,dy) = 4x^4 + 2tx^2y = \mathrm{d}_/ f \wedge (-y\,dx) = 4y^4 + tx^2y$$

we get

$$4x^4 = -\frac{8}{t^2}b(y^2) + b^2\mathbb{E}. \tag{5.6}$$

Therefore

$$4y^4 = b(1) - tx^2y = \frac{1}{2}b(1) - \frac{4}{t^2}b(y^2) + b^2\mathbb{E}. \tag{5.7}$$

The relation

$$x^4y^2 = \mathrm{d}_/ f \wedge \frac{x^3y\,dy + 4x^4y\,dx}{2\delta}$$

deduced from (5.3) shows $x^4y^2 \in b^2\mathbb{E}$.

Relation (5.4) rewritten as $2t^2x^2y = tb(1) + 16x^2y^3$ yields

$$x^2y^3 = \frac{t^2}{8}x^2y - \frac{t}{16}b(1).$$

Moreover

$$y^3(4y^3 + tx^2) = y^3\frac{\partial f}{\partial y} = \mathrm{d}_/ f \wedge (-y^3\,dx) = 3b(y^2)$$

and hence

$$4y^6 = -tx^2y^3 + 3b(y^2).$$

On the other hand

$$b(y^2) = \mathrm{d}_/ f \wedge (xy^2\,dy) = 4x^4y^2 + 2tx^2y^3 = 2tx^2y^3 + b^2\mathbb{E}.$$

Finally

$$a(y^2) = x^4y^2 + y^6 + tx^2y^3 - \frac{t}{4}x^2y^3 + \frac{3}{4}b(y^2) + tx^2y^3 + b^2\mathbb{E} = \frac{9}{8}b(y^2) + b^2\mathbb{E}.$$

Now, after (5.6), (5.7) and (5.5) we obtain successively

$$\begin{aligned} a(1) &= x^4 + y^4 + tx^2y \\ &= -\frac{2}{t^2}b(y^2) + \frac{1}{8}b(1) - \frac{1}{t^2}b(y^2) + \frac{1}{2}b(1) + \frac{4}{t^2}b(y^2) + b^2\mathbb{E} \\ &= \frac{5}{8}b(1) + \frac{1}{t^2}b(y^2) + b^2\mathbb{E}, \end{aligned}$$

$$a\left(1 - \frac{2y^2}{t^2}\right) = \frac{5}{8}b(1) + \frac{1}{t^2}b(y^2) - \frac{2}{t^2}\frac{9}{8}b(y^2) + b^2\mathbb{E} = \frac{5}{8}b\left(1 - \frac{2y^2}{t^2}\right) + b^2\mathbb{E}.$$

Some more computations of the same type left to the reader give

$$\begin{aligned} a(x) &= b(x) + b^2\mathbb{E}, \\ a(y) &= \frac{7}{8}b(y) + b^2\mathbb{E}, \\ a(x^2) &= \frac{11}{8}b(x^2) + b^2\mathbb{E}. \end{aligned}$$

Let us compute the monodromy M of t on the eigenvector $v_0 := 1 - \frac{2y^2}{t^2} + b\mathbb{E}$. Because $b^{-1}\nabla = \frac{\partial}{\partial t}$ it is given by

$$M = \exp(2i\pi t b^{-1}\nabla).$$

We have

$$\nabla(1) = -x^2y = -b\left(\frac{1}{2t}1 + \frac{4y^2}{t^3}\right) + b^2\mathbb{E}$$

and hence

$$t\frac{\partial}{\partial t}(1) = \frac{-1}{2}1 - \frac{4y^2}{t^2} + b\mathbb{E}.$$

Also

$$\nabla(y^2) = -x^2y^3 = -\frac{t^2}{8}x^2y + \frac{t}{16}b(1) = -\frac{t^2}{8}\left(\frac{1}{2t}b(1) + \frac{4}{t^3}b(y^2)\right) + \frac{t}{16}b(1) + b^2\mathbb{E}$$

gives

$$t\frac{\partial}{\partial t}(y^2) = -\frac{1}{2}y^2 + b\mathbb{E}.$$

Hence

$$t\frac{\partial}{\partial t}\left(1 - \frac{2y^2}{t^2}\right) - \frac{1}{2}\left(1 - \frac{2y^2}{t^2}\right) + b\mathbb{E}$$

from what we deduce $Mv = -v$.

An analogous computation with $\tau^2 = t$, shows that the eigenvector $\tilde{v}$ is invariant under $\tilde{M}$. On the other hand, the relation

$$\tilde{v} = 1 - \frac{2y^2}{\tau^4} + b\tilde{\mathbb{E}}$$

where $\tilde{\mathbb{E}}$ is associated to the pair $(\tilde{f}, \tau)$, with $\tilde{f}(x, y, \tau) := x^4 + y^4 + \tau^2x^2y$, shows that $\tilde{v}$ does not extend through 0 as a section of $\tilde{\mathbb{E}}$.

This last assertion may be proved directly. It suffices to show that there does not exist a holomorphic non-trival[2] 1-form $\tilde{\omega}$ near 0 such that

$$\mathrm{d}\tilde{\omega} = \frac{5}{8}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\omega}. \tag{5.8}$$

Because $\tilde{f}$ is quasi-homogeneous of degree 8 with the weights $(2,2,1)$ and because $\tilde{\omega}/\tilde{f}^{5/8}$ is homogeneous of degree 0, the form $\tilde{\omega}$ must be homogeneous of degree 5. So we may write

$$\tilde{\omega} = (\alpha_0 + \alpha_1\tau^2 + \alpha_2\tau^4)d\tau + \tau\beta_0 + \tau^3\beta_1$$

where α_j and β_j are respectively 0- and 1-homogeneous forms of degree $2-j$ with respect to x, y. Setting $\beta := \beta_0 + \tau^2\beta_1$, we get

$$\mathrm{d}\tilde{f} \wedge \tilde{\omega} = \mathrm{d}_/\tilde{f} \wedge \tau\beta \quad \text{and} \quad \mathrm{d}\tilde{\omega} = \tau d_/\beta \quad \text{modulo} \quad d\tau \wedge \square.$$

With (5.8) we deduce

$$8\tilde{f}\,\mathrm{d}_/\beta = 5\,\mathrm{d}_/\tilde{f} \wedge \beta$$

and an easy computation shows that this can hold only if $\beta = 0$. In that case $\alpha = 0$ also and the assertion follows. $\square$

Remark 5.3. It is easy to check that in the previous example the holomorphic 2-form

$$\tilde{\Omega} := 2x\,dy \wedge d\tau + 2y\,d\tau \wedge dx + \tau\,dx \wedge dy$$

satisfies

$$\mathrm{d}\tilde{\Omega} = \frac{5}{8}\frac{\mathrm{d}\tilde{f}}{\tilde{f}} \wedge \tilde{\Omega}$$

but it is not as easy as in Example 5.1 to see that $\tilde{ob}_1([\tilde{\Omega}])$ is not an extendable section of $\tilde{H}^1(5/8)$ or that $\theta(\tilde{\Omega})$ is not 0 in $H^1(\tilde{S}^*, \tilde{H}(5/8))$. The reason is that the transversal singularity is much more complicated in this example, and the development above consists precisely to compute the section $\tilde{ob}_1([\tilde{\Omega}])$ on $\tilde{S}^*$ of the sheaf $\tilde{H}(5/8)$ induced by $[\tilde{\Omega}]$ and to see that it does not extend at the origin which means, with the notations of [4], that $\tilde{ob}_1([\tilde{\Omega}]) \neq 0$ in $H^1_{\{0\}}(\tilde{S}, \tilde{H}(5/8))$.

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On Involutive Systems of First-order Nonlinear Partial Differential Equations

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Dedicated to Linda Rothschild

Abstract. We study the local and microlocal analyticity of solutions u of a system of nonlinear pdes of the form

$$F_j(x, u, u_x) = 0, \quad 1 \leq j \leq n$$

where the $F_j(x, \zeta_0, \zeta)$ are complex-valued, real analytic in an open subset of $(x, \zeta_0, \zeta) \in \mathbb{R}^N \times \mathbb{C} \times \mathbb{C}^N$ and holomorphic in (ζ_0, ζ). The function u is assumed to be a C^2 solution. The F_j satisfy an involution condition and $d_\zeta F_1 \wedge \cdots \wedge d_\zeta F_n \neq 0$ where d_ζ denotes the exterior derivative in $\zeta = (\zeta_1, \ldots, \zeta_N)$.

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0. Introduction

In this article we will study the local and microlocal regularity of solutions of certain overdetermined systems of first-order pdes of the form

$$F_j(x, u, u_x) = 0, \quad 1 \leq j \leq n$$

which are involutive. For some of the results we will only assume that u is a solution in some wedge $\mathcal{W}$. As an application of our results, in Example 2.9 we will study the analyticity of solutions u of the nonlinear system

$$\frac{\partial u}{\partial \overline{z}_k} - i\frac{\partial \phi}{\partial \overline{z}_k}\frac{\partial u}{\partial s} = a_k \left(\frac{\partial u}{\partial s}\right)^m + f_k, \quad 1 \leq k \leq n,$$

where $m \geq 2$ is any integer, $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and s are coordinates in $\mathbb{R}^{2n+1}$ and $\phi = \phi(x, y)$, $f_k(x, y, s)$ are real analytic functions.

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The nonlinear systems considered here are generalizations of the linear case where one considers a pair $(\mathcal{M},\mathcal{V})$ in which $\mathcal{M}$ is a manifold, and $\mathcal{V}$ is a subbundle of the complexified tangent bundle $\mathbb{C}T\mathcal{M}$ which is involutive, that is, the bracket of two sections of $\mathcal{V}$ is also a section of $\mathcal{V}$. We will refer to the pair $(\mathcal{M},\mathcal{V})$ as an involutive structure. The involutive structure $(\mathcal{M},\mathcal{V})$ is called locally integrable if the orthogonal of $\mathcal{V}$ in $\mathbb{C}T^*\mathcal{M}$ is locally generated by exact forms. A function (or distribution) u is called a solution of the involutive structure $(\mathcal{M},\mathcal{V})$ if $Lu=0$ for every smooth section L of $\mathcal{V}$. When $(\mathcal{M},\mathcal{V})$ is locally integrable, the local and microlocal regularity of solutions u has been studied extensively. Some of these results were extended in [EG] where regularity results were proved for boundary values of solutions defined in a wedge $\mathcal{W}$ in $\mathcal{M}$. Microlocal regularity of solutions for a single nonlinear equation in the C^∞ or analytic cases was investigated in the papers [A], [B], [Che], and [HT]. For results on the microlocal analytic regularity of solutions of higher-order linear differential operators, we mention the works [H1] and [H2].

When $(\mathcal{M},\mathcal{V})$ is an involutive structure, near a point $p\in\mathcal{M}$, one can always choose local coordinates (x,t), $x=(x_1,\dots,x_m)$, $t=(t_1,\dots,t_n)$, $m+n=\dim\mathcal{M}$ and smooth vector fields

$$L_j=\frac{\partial}{\partial t_j}+\sum_{k=1}^{m}a_{jk}(x,t)\frac{\partial}{\partial x_k},\quad 1\le j\le n,$$

such that $\{L_1,\dots,L_n\}$ form a basis of $\mathcal{V}$ on some neighborhood U of p. A distribution $u\in\mathscr{D}'(U)$ is a solution if

$$L_ju=0,\quad 1\le j\le n. \tag{0.1}$$

Likewise, for the nonlinear system $F_j(x,u,u_x)=0,\quad 1\le j\le n$ studied here, we can choose local coordinates $(x,t)\in\mathbb{R}^m\times\mathbb{R}^n$ so that the equations take the form

$$u_{t_j}=f_j(x,t,u_x),\quad 1\le j\le n.$$

Complex-valued solutions of first-order nonlinear pdes arise in several geometric and physical situations (see [CCCF], [KO1], and [KO2]).
This article is organized as follows. In section 1 we introduce involutive systems of first-order nonlinear pdes. The reader is referred to [T1] for a more detailed treatment of the subject. In the same section we will recall some concepts from [EG] that we will need to state our results. Section 2 contains the statements of our results and some examples. In particular, Example 2.9 shows a class of systems of involutive nonlinear pdes where the solutions are always real analytic. Sections 3 and 4 are devoted to the proofs of these results.

1. Preliminaries

The complex one-jet bundle of a smooth manifold $\mathcal{M}$ will be denoted by $\mathbb{C}J^1(\mathcal{M})$. An element of $\mathbb{C}J^1(\mathcal{M})$ is a triple (x,a,ω) where $x\in\mathcal{M}$, $a\in\mathbb{C}$, and $\omega\in\mathbb{C}T^*_x\mathcal{M}$. We can therefore identify $\mathbb{C}J^1(\mathcal{M})$ with $\mathbb{C}\times\mathbb{C}T^*\mathcal{M}$. Let $\dim\mathcal{M}=N$ and suppose

U is the domain of local coordinates $x_1, \ldots, x_N$. Let $\zeta_1, \ldots, \zeta_N$ denote the corresponding complex coordinates in $\mathbb{C}T_x^*\mathcal{M}$ at $x \in U$. Let $\mathcal{O} = \mathbb{C} \times (\mathbb{C}T^*\mathcal{M}|_U) \cong U \times \mathbb{C}^{N+1}$ be the open subset of the one-jet bundle that lies over U. The coordinate in the first factor of $\mathbb{C}^{N+1}$ will be denoted by ζ_0.

Let $F_1(x, \zeta_0, \zeta)$, …, $F_n(x, \zeta_0, \zeta)$ be n C^∞ functions on $\mathcal{O}$ that are holomorphic in (ζ_0, ζ). We will be studying systems of pdes of the form

$$F_j(x, u, u_x) = 0, \quad 1 \le j \le n \tag{1.1}$$

generalizing the linear system (0.1). The linear independence of the system in (0.1) is generalized by assuming that

$$d_\zeta F_1 \wedge \cdots \wedge d_\zeta F_n \neq 0 \quad \text{at every point of } \mathcal{O}. \tag{1.2}$$

Here d_ζ denotes the exterior derivative with respect to $(\zeta_1, \ldots, \zeta_n)$. Condition (1.2) implies that $n \le N$. Moreover, if there is a point $p \in \mathcal{O}$ where $F_1(p) = \cdots = F_n(p) = 0$, then the set

$$\Sigma = \{(x, \zeta_0, \zeta) \in \mathcal{O} : \; F_j(x, \zeta_0, \zeta) = 0, \; 1 \le j \le n\} \tag{1.3}$$

is a smooth submanifold of $\mathcal{O}$ whose intersection with each fiber $\mathbb{C}^{N+1}$ is a holomorphic submanifold of complex dimension $N + 1 - n$.

The involutivity of the linear system (0.1) is generalized by using a notion of Poisson brackets which we will now describe. If $F = F(x, \zeta_0, \zeta)$ is a smooth function on $\mathcal{O}$ which is holomorphic in (ζ_0, ζ), we define the holomorphic Hamiltonian of F by

$$\begin{aligned} H_F = \sum_{i=1}^N \frac{\partial F}{\partial \zeta_i}\frac{\partial}{\partial x_i} - \sum_{i=1}^N \left(\frac{\partial F}{\partial x_i} + \zeta_i \frac{\partial F}{\partial \zeta_0}\right)\frac{\partial}{\partial \zeta_i} \\ + \left[\sum_{i=1}^N \zeta_i \frac{\partial F}{\partial \zeta_i} - F\right]\frac{\partial}{\partial \zeta_0} + \frac{\partial F}{\partial \zeta_0}. \end{aligned} \tag{1.4}$$

If $G = G(x, \zeta_0, \zeta)$ is also a similar function, we define the Poisson bracket $\{F, G\}$ by

$$\{F, G\} = H_F G = -H_G F. \tag{1.5}$$

Observe that for the class of functions being considered, the definition of the Poisson bracket is independent of the choice of local coordinates $x_1, \ldots, x_N$ since this is true for each of the three vector fields

$$\sum_{i=1}^N \frac{\partial F}{\partial \zeta_i}\frac{\partial}{\partial x_i} - \sum_{i=1}^N \frac{\partial F}{\partial x_i}\frac{\partial}{\partial \zeta_i}, \quad \sum_{i=1}^N \zeta_i \frac{\partial}{\partial \zeta_i}, \text{ and } \frac{\partial}{\partial \zeta_0}.$$

Going back to the equations (1.1), the involutivity condition can be expressed as:

$$\{F_j, F_k\} = 0 \text{ on the set (1.3) } \; \forall\, j, k = 1, \ldots, n. \tag{1.6}$$

Condition (1.6) is a formal integrability condition for the system of equations $F_j(x, u, u_x) = 0$, $j = 1, \ldots, n$. Indeed, suppose for every (x_0, ζ_0', ζ') in the set Σ of (1.3), there is a C^2 solution u of the equations $F_j(x, u, u_x) = 0$, $j = 1, \ldots, n$

near the point x_0 satisfying $u(x_0) = \zeta_0'$, and $u_x(x_0) = \zeta'$. Let $F = F_m$, and $G = F_k$ for some $m, k \in \{1, \dots, n\}$. Differentiating the equations $F(x, u, u_x) = 0$ and $G(x, u, u_x) = 0$ we get:

$$F_{x_i} + F_{\zeta_0} u_{x_i} + \sum_{j=1}^{N} F_{\zeta_j} u_{x_j x_i} = 0, \ 1 \le i \le n \tag{1.7}$$

at the points $(x, u(x), u_x(x))$, and likewise,

$$G_{x_i} + G_{\zeta_0} u_{x_i} + \sum_{j=1}^{N} G_{\zeta_j} u_{x_j x_i} = 0, \ 1 \le i \le n. \tag{1.8}$$

Multiply (1.7) by $-G_{\zeta_i}$, (1.8) by F_{ζ_i} and add over i to get:

$$\sum_{i=1}^{N} F_{\zeta_i}(G_{x_i} + G_{\zeta_0} u_{x_i}) - \sum_{i=1}^{N} G_{\zeta_i}(F_{x_i} + F_{\zeta_0} u_{x_i}) = \sum_{i=1}^{N} \sum_{j=1}^{N} (G_{\zeta_i} F_{\zeta_j} - G_{\zeta_j} F_{\zeta_i}) u_{x_j x_i}.$$

The right-hand side of the preceding equation is clearly zero and hence, at the points $(x, u(x), u_x(x))$, we get

$$\sum_{i=1}^{N} F_{\zeta_i}(G_{x_i} + G_{\zeta_0} u_{x_i}) - \sum_{i=1}^{N} G_{\zeta_i}(F_{x_i} + F_{\zeta_0} u_{x_i}) = 0.$$

Since $(x, u(x), u_x(x) \in \Sigma$, the latter equation implies that at such points,

$$\sum_{i=1}^{N} F_{\zeta_i}(G_{x_i} + G_{\zeta_0} u_{x_i}) - \sum_{i=1}^{N} G_{\zeta_i}(F_{x_i} + F_{\zeta_0} u_{x_i}) + F_{\zeta_0} G - F G_{\zeta_0} = 0,$$

that is, $H_F G = \{F, G\} = 0$ on Σ.

A global definition of involutive systems can now be stated as follows:

Definition 1.1. An involutive system of first-order partial differential equations of rank n on $\mathcal{M}$ is a closed C^∞ submanifold Σ of $\mathbb{C}J^1(\mathcal{M})$ satisfying the following properties:

(i) the projection $\mathbb{C}J^1(\mathcal{M}) \to \mathcal{M}$ maps Σ onto $\mathcal{M}$;

(ii) each point of Σ has a neighborhood $\mathcal{O}$ on which there are n C^∞ functions $F_1(x, \zeta_0, \zeta), \dots, F_n(x, \zeta_0, \zeta)$, holomorphic with respect to (ζ_0, ζ), such that $\Sigma \cap \mathcal{O}$ is the set (1.3), and that (1.2) and (1.6) hold.

Definition 1.2. Let Σ be as in Definition 1.1. A C^1 function u on $\mathcal{M}$ is called a solution if its first jet lies in Σ.

Example 1.3. Let $(\mathcal{M}, \mathcal{V})$ be an involutive structure. Let $\mathcal{V}^\perp \subseteq \mathbb{C}T^*\mathcal{M}$ denote the one-forms ω such that $\langle \omega, L \rangle = 0$ for every section L of $\mathcal{V}$. Let Σ be the preimage of $\mathcal{V}^\perp$ under the projection $\mathbb{C}J^1(\mathcal{M}) \to \mathbb{C}T^*\mathcal{M}$. Let $\{L_1, \dots, L_n\}$ be a smooth basis of $\mathcal{V}$ over an open set U such that $[L_i, L_j] = 0 \ \forall \ i, j = 1, \dots, n$. Then, over U, Σ is defined by the vanishing of the symbols $F_j(x, \zeta)$ ($\zeta \in \mathbb{C}^N$) of the L_j regarded as functions on $\mathbb{C}J^1(\mathcal{M})|_U$. Property (1.2) follows from the linear independence of the L_j and (1.6) is a consequence of the fact that the L_j commute. Thus every involutive structure $(\mathcal{M}, \mathcal{V})$ defines an involutive system of first-order partial differential equations on $\mathcal{M}$.

We will next recall some notions on involutive structures that we will need to state our results. The reader is referred to [EG] for more details.

Let $(\mathcal{M}, \mathcal{V})$ be a smooth involutive structure with fiber dimension of $\mathcal{V}$ over $\mathbb{C}$ equal to n.

Definition 1.4. A smooth submanifold X of $\mathcal{M}$ is called maximally real if

$$\mathbb{C}T_p\mathcal{M} = \mathcal{V}_p \oplus \mathbb{C}T_pX \quad \text{for each } p \in X.$$

Note that if $(\mathcal{M}, \mathcal{V})$ is CR, then X is maximally real if and only if it is totally real of maximal dimension.

If X is a maximally real submanifold and $p \in X$, define

$$\mathcal{V}_p^X = \{L \in \mathcal{V}_p : \Re L \in T_pX\}.$$

We recall the following result from [EG]:

Proposition 1.5. (*Lemma* II.1 *in* [EG]) *$\mathcal{V}^X$ is a real subbundle of $\mathcal{V}|_X$ of rank n. The map*

$$\Im : \mathcal{V}^X \to T\mathcal{M}$$

which takes the imaginary part induces an isomorphism

$$\mathcal{V}^X \cong T\mathcal{M}|_X / TX.$$

Proposition 1.5 shows that when X is maximally real, for $p \in X$, $\Im$ defines an isomorphism from $\mathcal{V}_p^X$ to an n-dimensional subspace N_p of $T_p\mathcal{M}$ which is a canonical complement to T_pX in the sense that

$$T_p\mathcal{M} = T_pX \oplus N_p.$$

Definition 1.6. Let E be a submanifold of $\mathcal{M}$, $\dim E = k$. We say an open set $\mathcal{W}$ is a wedge in $\mathcal{M}$ at $p \in E$ with edge E if the following holds: there exists a diffeomorphism F of a neighborhood V of 0 in $\mathbb{R}^N$ ($N = \dim \mathcal{M}$) onto a neighborhood U of p in $\mathcal{M}$ and a set $B \times \Gamma \subset V$ with B a ball centered at $0 \in \mathbb{R}^k$ and Γ a truncated, open convex cone in $\mathbb{R}^{N-k}$ with vertex at 0 such that

$$F(B \times \Gamma) = \mathcal{W} \quad \text{and} \quad F(B \times \{0\}) = E \cap U.$$

Definition 1.7. Let E, $\mathcal{W}$ and $p \in E$ be as in Definition 1.6. The direction wedge $\Gamma_p(\mathcal{W}) \subset T_p\mathcal{M}$ is defined as the interior of the set

$$\{c'(0) |\ c : [0, 1) \to \mathcal{M} \text{ is } C^\infty,\ c(0) = p,\ c(t) \in \mathcal{W}\ \forall\ t > 0\}.$$

Observe that $\Gamma_p(\mathcal{W})$ is a linear wedge in $T_p\mathcal{M}$ with edge T_pE. Set

$$\Gamma(\mathcal{W}) = \bigcup_{p \in E} \Gamma_p(\mathcal{W})$$

Suppose $\mathcal{W}$ is a wedge in $\mathcal{M}$ with a maximally real edge X. As observed in [EG], since $\Gamma_p(\mathcal{W})$ is determined by its image in $T_p\mathcal{M}/T_pX$, the isomorphism $\Im$ can be used to define a corresponding wedge in $\mathcal{V}_p^X$ by setting

$$\Gamma_p^{\mathcal{V}}(\mathcal{W}) = \{L \in \mathcal{V}_p^X : \Im L \in \Gamma_p(\mathcal{W})\}.$$

$\Gamma_p^{\mathcal{V}}(\mathcal{W})$ is a linear wedge in $\mathcal{V}_p^X$ with edge $\{0\}$, that is, it is a cone. Define next

$$\Gamma_p^T(\mathcal{W}) = \{\Re L : L \in \Gamma_p^{\mathcal{V}}(\mathcal{W})\}.$$

Since the map $\Re : \mathcal{V}_p^X \to \Re\mathcal{V}_p \cap T_pX$ is onto, $\Gamma_p^T(\mathcal{W})$ is an open cone in $\Re\mathcal{V}_p \cap T_pX$. Let

$$\Gamma^T(\mathcal{W}) = \bigcup_{p\in X} \Gamma_p^T(\mathcal{W}).$$

2. Main results and examples

In this section we will assume that $\mathcal{M}$ is a real analytic manifold and we will consider a real analytic involutive system Σ by which we mean that the local defining functions F_j in Definition 1.1 are real analytic. A real analytic wedge $\mathcal{W}$ in $\mathcal{M}$ is one where the diffeomorphism F in Definition 1.6 is real analytic. Suppose u is a C^2 solution of the involutive system Σ on an open set $U \subseteq \mathcal{M}$ which is a domain of local coordinates $x_1, \ldots, x_N$. Let $\mathcal{O}$ be an open set in $\mathbb{C}J^1(\mathcal{M})|_U$ and $F_1(x, \zeta_0, \zeta), \ldots, F_n(x, \zeta_0, \zeta)$ functions in $\mathcal{O}$ satisfying the conditions of Definition 1.1. Consider the vector fields on U defined by

$$L_j^u = \sum_{k=1}^N \frac{\partial F_j}{\partial \zeta_k}(x, u(x), u_x(x)) \frac{\partial}{\partial x_k}, \quad 1 \le j \le n.$$

If for $v \in C^1(U)$ we define F_j^v by

$$F_j^v(x) = F_j(x, v(x), v_x(x)),$$

then we see that $L_j^u(v)$ is the principal part of the Fréchet derivative of the map

$$v \mapsto F_j^v$$

at u. We will refer to the vector fields L_j^u as the linearized operators of $F_j(x, u(x), u_x(x)) = 0$ at u. Since the F_j satisfy (1.2), $L_1^u, \ldots, L_n^u$ are linearly independent and span a bundle $\mathcal{V}^u$ over U. If $G_1(x, \zeta_0, \zeta), \ldots, G_n(x, \zeta_0, \zeta)$ also satisfy Definition 1.1, then each $G_k = \sum_{j=1}^n a_{ij} F_j$ for some $a_{ij} \in C^\infty(\mathcal{O})$ and so the linearized operators of $G_k(x, u(x), u_x(x)) = 0$ at u are linear combinations of the L_j^u. In Section 3 we will see that $\mathcal{V}^u$ is closed under brackets.

In what follows, $WF_a u$ denotes the analytic wave front set of a function u.

The main results of this article can be stated as follows:

Theorem 2.1. *Let Σ be a real analytic involutive system of first-order partial differential equations of rank n on a real analytic manifold $\mathcal{M}$. Let $\mathcal{W}$ be a real analytic wedge in $\mathcal{M}$ with a real analytic edge E. Suppose $u \in C^2(\overline{\mathcal{W}})$ and it is a solution on $\mathcal{W}$. Assume that*

$$\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_pE \oplus \mathcal{V}_p^u \quad \forall\, p \in E.$$

Then $WF_a(u_0) \subseteq (\Gamma^T(\mathcal{W}))^\circ$, where $u_0 = u|_E$, and polar refers to the duality between TE and T^*E.

Theorem 2.1 will be proved in Section 3. In the linear case, Theorem 2.1 was proved in [EG].

Given an involutive structure $(\mathcal{M}, \mathcal{V})$, for $p \in \mathcal{M}$, let $T_p^\circ(\mathcal{M}) = \mathcal{V}_p^\perp \cap T_p^*(\mathcal{M})$ which is the characteristic set of $\mathcal{V}$ at p. If $\mathcal{W}$ is a wedge in $\mathcal{M}$ with edge E, recall that $\Gamma_p^T(\mathcal{W})$ is an open cone in $\Re\mathcal{V}_p \cap T_pE$.
Since $\Re\mathcal{V}_p \cap T_pE = \left(\iota_E^*(T_p^\circ\mathcal{M})\right)^\perp$, where $\iota_E^* : T_p^*(\mathcal{M}) \to T_p^*(E)$ is the pullback map, we have:

$$\iota_E^*(T_p^\circ\mathcal{M}) \subseteq \left(\Gamma_p^T(\mathcal{W})\right)^\circ.$$

As in the linear case (see [EG]), if in the nonlinear context of Theorem 2.1 $u_0 = u|_E$ is the trace of two solutions defined in opposite wedges, we get the following stronger conclusion:

Corollary 2.2. *Let Σ be a real analytic involutive system of first-order partial differential equations of rank n on a real analytic manifold $\mathcal{M}$. Let $\mathcal{W}^+$ and $\mathcal{W}^-$ be real analytic wedges in $\mathcal{M}$ with a real analytic edge E having opposite directions at $p \in E$, that is $\Gamma_p(\mathcal{W}^+) = -\Gamma_p(\mathcal{W}^-)$. Let $u^+ \in C^2(\overline{\mathcal{W}^+})$, $u^- \in C^2(\overline{\mathcal{W}^-})$ be solutions in $\mathcal{W}^+$ and $\mathcal{W}^-$ respectively such that $u^+|_E = u_0 = u^-|_E$, and assume that $\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_pE \oplus \mathcal{V}_p$ (where $\mathcal{V}_p = \mathcal{V}_p^{u^+} = \mathcal{V}_p^{u^-}$). Then, $WF_a(u_0)|_p \subseteq \iota_E^*(T_p^\circ(\mathcal{M}))$.*

Proof. By Theorem 2.1 we know that $WF_a(u_0) \subseteq (\Gamma_p^T(\mathcal{W}^+))^\circ \cap (\Gamma_p^T(\mathcal{W}^-))^\circ$. We also always have

$$\iota_E^*(T_p^\circ(\mathcal{M})) \subseteq (\Gamma_p^T(\mathcal{W}^+))^\circ \cap (\Gamma_p^T(\mathcal{W}^-))^\circ.$$

If $\sigma \in (\Gamma_p^T(\mathcal{W}^+))^\circ \cap (\Gamma_p^T(\mathcal{W}^-))^\circ$, then $\langle \sigma, v\rangle = 0\ \forall\ v \in \Gamma_p^T(\mathcal{W}^+)$. But then since $\Gamma_p^T(\mathcal{W}^+)$ is open in $(\mathrm{Re}\ \mathcal{V}_p) \cap T_pE$, σ has to vanish on $\Re\mathcal{V}_p \cap T_pE = (\iota_E^*(T_p^\circ\mathcal{M}))^\perp$ and hence $\sigma \in \iota_E^*(T_p^\circ\mathcal{M})$. Thus,

$$WF_a(u_0) \subseteq \iota_E^*(T_p^\circ\mathcal{M}) \qquad \square$$

In Section 3 we will show that Corollary 2.2 implies a generalization of the classical edge of the wedge theorem for nonlinear systems of pdes (see Corollary 3.6).

When u is a solution in a full neighborhood, we get:

Corollary 2.3. *Let Σ be a real analytic involutive system of first-order partial differential equations of rank n on a real analytic manifold $\mathcal{M}$. Let u be a C^2 solution. Then*

$$WF_a u \subseteq (\mathcal{V}^u)^\perp \cap T^*\mathcal{M} = T^\circ\mathcal{M}.$$

Proof. We will use a trick from [HT]. We may assume that in a neighborhood of the origin in $\mathbb{R}^N$, u satisfies the system of equations $F_j(x, u, u_x) = 0$ for $j = 1, \dots, n$ where the F_j are real analytic and satisfy the conditions in Definition 1.1. Introduce additional variables $y_1, \dots, y_n$ with respective dual coordinates $\eta_1, \dots, \eta_n$ and for each $\theta \in [0, 2\pi)$, define

$$G_j^\theta(x, y, \zeta_0, \zeta, \eta) = \eta_j - e^{i\theta} F_j(x, \zeta_0, \zeta),\ 1 \le j \le n.$$

Let $w(x, y) = u(x)$. Note that $G_j^\theta(x, y, w, w_x, w_y) = 0$ for all j. It is easy to see that the G_j^θ satisfy conditions (1.2) and (1.6). The linearized operators at w are:

$$L_{\theta,j}^w = \frac{\partial}{\partial y_j} - e^{i\theta} \sum_{k=1}^N \frac{\partial F_j}{\partial \zeta_k}(x, u(x), u_x(x)) \frac{\partial}{\partial x_k}, \quad 1 \le j \le n.$$

Let $\widetilde{\mathcal{M}} = \mathbb{R}_x^N \times \mathbb{R}_y^n$ and $E = \mathbb{R}_x^N \times \{0\}$. Note that the characteristic set of the $L_{\theta,j}^w$ at the origin in $\widetilde{\mathcal{M}}$ is

$$T_\theta^0 \widetilde{\mathcal{M}} = \{(0; \xi, \eta) : \eta_j = e^{i\theta} D_\zeta F_j(0, u(0), u_x(0)) \cdot \xi,\ j = 1, \dots, n\}.$$

It follows that

$$\iota_E^*(T_\theta^0 \widetilde{\mathcal{M}}) = \{(0; \xi) : e^{i\theta} D_\zeta F_j(0, u(0), u_x(0)) \cdot \xi \text{ is real for } j = 1, \dots, n\}.$$

Observe that at the origin in $\mathcal{M}$, the characteristic set of the operators

$$L_j^u = \sum_{k=1}^N \frac{\partial F_j}{\partial \zeta_k}(x, u(x), u_x(x)) \frac{\partial}{\partial x_k}, \quad 1 \le j \le n$$

equals $\bigcap_{0 \le \theta \le 2\pi} \iota_E^*(T_\theta^0 \widetilde{\mathcal{M}})$. It is easy to see that $\mathbb{C}T_0\widetilde{\mathcal{M}} = \mathbb{C}T_0 E \oplus \mathcal{V}_\theta^w$ where $\mathcal{V}_\theta^w$ is the span of the $L_{\theta,j}^w$ at the origin. Let Γ^+ be any open cone in $\mathbb{R}_y^n$ and set $\Gamma^- = -\Gamma^+$. Then $u(x)$ is the common boundary value of the solution $w(x, y)$ of the G_j^θ defined in opposite wedges and hence by Corollary 2.2, $WF_a(u) \subset T^0\mathcal{M}$. $\square$

When $n = 1$, Corollary 2.3 was proved in [HT]. Again when $n = 1$ and the pde is quasilinear, Theorem 2.1 was proved in [M].

We will next present a nonlinear wedge version of the classical Lewy extension theorem.

Theorem 2.4. *Let Σ be a real analytic involutive system of first-order partial differential equations of rank n on a real analytic manifold $\mathcal{M}$. Let $\mathcal{W}$ be a real analytic wedge in $\mathcal{M}$ with a real analytic edge E.*

Suppose $u \in C^2(\overline{\mathcal{W}})$ and it is a solution on $\mathcal{W}$, and assume that

$$\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_pE \oplus \mathcal{V}_p^u \quad \forall\, p \in E.$$

Let $\sigma \in (\mathcal{V}_p^u)^\perp \cap T_p^*\mathcal{M} = T_p^\circ\mathcal{M}$ for some $p \in E$. If L is a C^1 section of $\mathcal{V}^u$ defined near p in $\overline{\mathcal{W}}$ such that $L(p) \in \Gamma_p^{\mathcal{V}}(\mathcal{W})$ and

$$\frac{1}{i}\langle\sigma, [L, \bar{L}]\rangle < 0, \qquad \text{then } i_E^*\sigma \notin WF_a u_0 \quad (u_0 = u|_E).$$

When the solution u is defined in a full neighborhood of a point p, by adding variables as in the proof of Corollary 2.3, we get:

Corollary 2.5. *Let Σ be a real analytic involutive system of first-order partial differential equations of rank n on a real analytic manifold $\mathcal{M}$. Suppose u is a C^2 solution in a neighborhood of a point p. Let $\sigma \in (\mathcal{V}_p^u)^\perp \cap T_p^*\mathcal{M} = T_p^\circ\mathcal{M}$. If L is a C^1 section of $\mathcal{V}^u$ defined near p such that $\frac{1}{i}\langle\sigma, [L, \bar{L}]\rangle < 0$, then $\sigma \notin WF_a u$.*

Proof. We assume the point p is the origin of $\mathbb{R}^N$. Let the F_j, $\widetilde{\mathcal{M}}$, and E be as in the proof of Corollary 2.3. For each $1 \le j \le n$, let

$$G_j(x, y, \zeta_0, \zeta, \eta) = b_j\eta_j - F_j(x, \zeta_0, \zeta)$$

where the b_j are constants that will be chosen. If $w(x, y) = u(x)$, then it satisfies the equations $G_j(x, y, w, w_x, w_y) = 0$ for all j, and the linearizations at w are

$$L_j^w = b_j\frac{\partial}{\partial y_j} - \sum_{k=1}^N \frac{\partial F_j}{\partial \zeta_k}(x, u(x), u_x(x))\frac{\partial}{\partial x_k}, \quad 1 \le j \le n.$$

Let

$$L_j^u = \sum_{k=1}^N \frac{\partial F_j}{\partial \zeta_k}(x, u(x), u_x(x))\frac{\partial}{\partial x_k}, \quad 1 \le j \le n.$$

Let $L = \sum_{k=1}^n a_k L_k^u$ and define $L' = \sum_{k=1}^n a_k L_k^w$. We choose the b_j so that $\Re L'(0) \in T_0E$, and $\Im L'(0) \notin T_0E$. This is possible since $L(0) \neq 0$ because by hypothesis, $\langle\sigma, [L, \overline{L}]\rangle \neq 0$. Let $\mathcal{W}$ be a wedge in $\widetilde{\mathcal{M}}$ with edge E such that $L'(0) \in \Gamma_0^{\mathcal{V}^w}(\mathcal{W})$ where $\mathcal{V}^w$ is the bundle generated by the L_j^w. Let $\sigma' = (\sigma, 0)$. If $\iota_E^* : T_0^*\widetilde{\mathcal{M}} \to T_0^*E$ denotes the pullback map, $\iota^*\sigma' = \sigma$, and

$$\frac{1}{i}\langle\sigma', [L', \bar{L'}]\rangle = \frac{1}{i}\langle\sigma, [L, \bar{L}]\rangle < 0.$$

Hence, since $w|_E = u$, by Theorem 2.4, $\sigma \notin WF_a u$. □

In the linear case, Theorem 2.4 was proved in [EG]. When $n = 1$ and the pde is quasilinear, Theorem 2.4 was proved in [LMX] under a stronger assumption. For $n = 1$ Corollary 2.5 was obtained in [B].

We will indicate next why Theorem 2.2 in [LMX] is a special case of Theorem 2.4. We recall Theorem 2.2 in [LMX]. Consider the quasi-linear equation

$$\frac{\partial u}{\partial t}+\sum_{j=1}^{N}a_j(x,t,u)\frac{\partial u}{\partial x_j}=b(x,t,u)$$

on $\Omega\times[0,T)$ where Ω is an open subset of $\mathbb{R}^N$, and the functions $a_j, b, j=1,\dots,N$ are restrictions of holomorphic functions. Let

$$\mathcal{L}=\frac{\partial}{\partial t}+\sum_{j=1}^{N}a_j(x,t,\zeta_0)\frac{\partial}{\partial x_j}+b(x,t,\zeta_0)\frac{\partial}{\partial\zeta_0}.$$

Set $\nu_0=(a_1,\dots,a_N),\quad \nu_1=\mathcal{L}(\nu_0)=(\mathcal{L}(a_1),\dots,\mathcal{L}(a_N))$.

Theorem 2.6. (*Theorem* 2.2 *in* [LMX]). *Let u be a C^2 solution of the quasi-linear equation on $\Omega\times[0,T)$. If*

$$\forall x\in\Omega,\quad \Im\nu_0(x,0,u(x,0))=0,\quad \text{and}\quad \Im\nu_1(x_0,0,u(x_0),0)\cdot\xi>0,$$

then $(x_0,\xi)\notin WF_a u(x,0)$.

We will deduce this result from Theorem 2.4 by showing that whenever $\Im\nu_0(x_0,0,u(x_0,0))=0$, then

$$\Im\nu_1(x_0,0,u(x_0,0))\cdot\xi=-\frac{\langle(\xi,0),[L^u,\overline{L^u}](x_0,0)\rangle}{2i},$$

where we recall that

$$L^u=\frac{\partial}{\partial t}+\sum_{j=1}^{N}a_j(x,t,u)\frac{\partial u}{\partial x_j}.$$

Note that we will not require the vanishing of $\Im\nu_0(x,0,u(x,0))$ for all x. We have

$$-\frac{[L^u,\overline{L^u}]}{2i}=\Im\left(\sum_j\Big(\frac{\partial a_j}{\partial t}+\frac{\partial a_j}{\partial\zeta_0}\frac{\partial u}{\partial t}\Big)\frac{\partial}{\partial x_j}\right)$$
$$+\Im\left(\sum_j\sum_k\overline{a_k}\frac{\partial a_j}{\partial x_k}\frac{\partial}{\partial x_j}+\sum_j\sum_k\overline{a_k}\frac{\partial a_j}{\partial\zeta_0}\frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_j}\right).$$

Using the assumption that $\Im\nu_0(x_0,0,u(x_0,0))=0$ and the equation that u satisfies, we get

$$-\frac{[L^u,\overline{L^u}]}{2i}=\Im\left(\sum_j\Big(\frac{\partial a_j}{\partial t}+b\frac{\partial a_j}{\partial\zeta_0}\Big)\frac{\partial}{\partial x_j}\right)+\Im\left(\sum_j\sum_k a_k\frac{\partial a_j}{\partial x_k}\frac{\partial}{\partial x_j}\right)$$

at the point $(x_0,0)$. Thus for any $\xi\in\mathbb{R}^N$,

$$\Im\nu_1(x_0,0,u(x_0,0))\cdot\xi=-\frac{\langle(\xi,0),[L^u,\overline{L^u}](x_0,0)\rangle}{2i},$$

which shows that Theorem 2.6 follows from Theorem 2.4.

The following result generalizes the analogous linear result in [EG] (see [C] also).

Theorem 2.7. *Let Σ be a real analytic involutive system of first-order partial differential equations of rank n on a real analytic manifold $\mathcal{M}$. Let $\mathcal{W}$ be a real analytic wedge in $\mathcal{M}$ with a real analytic edge E. Suppose $u \in C^3(\overline{\mathcal{W}})$ and it is a solution on $\mathcal{W}$, and assume that*

$$\mathbb{C}T_p\mathcal{M} = \mathbb{C}T_pE \oplus \mathcal{V}_p^u \quad \forall\, p \in E.$$

Let $\sigma \in (\mathcal{V}_p^u)^\perp \cap T_p^*\mathcal{M} = T_p^\circ\mathcal{M}$ for some $p \in E$. If L is a C^2 section of $\mathcal{V}^u$ defined near p in $\overline{\mathcal{W}}$ such that $L(p) \in \Gamma_p^{\mathcal{V}}(\mathcal{W})$, $\langle\sigma, [L, \bar{L}]\rangle = 0$, and

$$\sqrt{3}|\Im\langle\sigma, [L, [L, \bar{L}]]\rangle| < \Re\langle\sigma, [L, [L, \bar{L}]]\rangle,$$

then $i_E^*\sigma \notin WF_a u_0 \quad (u_0 = u|_E)$.

When the solution u is defined in a full neighborhood of a point p, by adding variables as in the proof of Corollary 2.5, we get:

Corollary 2.8. *Let Σ be a real analytic involutive system of first-order partial differential equations of rank n on a real analytic manifold $\mathcal{M}$. Suppose u is a C^3 solution in a neighborhood of a point p. Let $\sigma \in (\mathcal{V}_p^u)^\perp \cap T_p^*\mathcal{M} = T_p^\circ\mathcal{M}$. If L is a C^2 section of $\mathcal{V}^u$ defined near p such that $\langle\sigma, [L, \bar{L}]\rangle = 0$, and*

$$\langle\sigma, [L, [L, \bar{L}]]\rangle \neq 0,$$

then $\sigma \notin WF_a u$.

Proof. We use the notations of Corollary 2.5. We have

$$\langle\sigma', [L', \bar{L'}]\rangle = \langle\sigma, [L, \bar{L}]\rangle = 0$$

and

$$\langle\sigma', [L', [L', \bar{L'}]]\rangle = \langle\sigma, [L, [L, \bar{L}]]\rangle \neq 0.$$

We can therefore choose $\theta \in [0, 2\pi)$ such that if $L'' = e^{i\theta}L'$, then

$$\sqrt{3}|\Im\langle\sigma', [L'', [L'', \bar{L''}]]\rangle| < \Re\langle\sigma, [L'', [L'', \bar{L''}]]\rangle,$$

and this inequality holds for any choice of the b_j. We now choose the b_j so that $\Re L''(0) \in T_0E$. Theorem 2.7 then implies that $\sigma \notin WF_a u$. □

Example 2.9. We consider the following nonlinear system in $\mathbb{R}^{2n+1}$ whose solvability when $m = 2$ was studied in [T2]. Let x_j, y_j $(j = 1, \ldots, n)$ and s denote the coordinates in $\mathbb{R}^{2n+1}$ and ξ_j, η_j, σ the respective dual coordinates. Let $m \geq 2$ be an integer. For $k = 1, \ldots, n$, let

$$p_k(x, y, s, \xi, \eta, \sigma) = \frac{1}{2}(\xi_k + i\eta_k) - a_k\sigma^m - ib_k(x, y)\sigma + f_k(x, y, s)$$

where $a_k \in \mathbb{C}$, $b_k(x, y)$ and $f_k(x, y, s)$ are real analytic functions. It is clear that $d_\xi p_1 \wedge \cdots \wedge d_\xi p_n \neq 0$ where $\xi = (\xi_1, \ldots, \xi_n)$. Assume that $a = (a_1, \ldots, a_n) \neq 0$. For each $k = 1, \ldots, n$, let

$$L_k = \frac{\partial}{\partial \bar{z}_k} - (ib_k(x, y) + ma_k\sigma^{m-1})\frac{\partial}{\partial s} \quad \text{and} \quad L_k^0 = \frac{\partial}{\partial \bar{z}_k} - ib_k(x, y)\frac{\partial}{\partial s}.$$

We have

$$H_{p_k} = \frac{\partial}{\partial \overline{z_k}} - (ma_k\sigma^{m-1} + ib_k)\frac{\partial}{\partial s} - \sum_{i=1}^{n} \frac{\partial p_k}{\partial x_i}\frac{\partial}{\partial \xi_i} - \sum_{i=1}^{n} \frac{\partial p_k}{\partial y_i}\frac{\partial}{\partial \eta_i} - \frac{\partial f_k}{\partial s}\frac{\partial}{\partial \sigma},$$

and therefore,

$$H_{p_k}p_j = i\sigma\left(\frac{\partial b_k}{\partial \overline{z}_j} - \frac{\partial b_j}{\partial \overline{z}_k}\right) + L_k f_j - L_j f_k.$$

Hence, the conditions:

$$(1) \quad \frac{\partial b_k}{\partial \overline{z}_j} = \frac{\partial b_j}{\partial \overline{z}_k} \quad \text{and} \quad (2) \quad L_k f_j = L_j f_k$$

imply the validity of the involution condition $\{p_j, p_k\} = 0$. To guarantee (1), we will assume that for some real analytic $\phi(x,y)$, $\frac{\partial \phi}{\partial \overline{z}_j} = b_j$ for $j = 1, \ldots, n$. Condition (2) is equivalent to the pair of equations

$$(3) \quad L_k^0 f_j = L_j^0 f_k \quad \text{and} \quad (4) \quad a_k\frac{\partial f_j}{\partial s} = a_j\frac{\partial f_k}{\partial s}.$$

Since $a \neq 0$, (4) is equivalent to the existence of a real analytic function $W(x,y,s)$ such that

$$(5) \quad f_j(x,y,s) = f_j(x,y,0) + a_j W(x,y,s),\ j = 1, \ldots, n.$$

Observe that once (5) holds, condition (3) can be satisfied if for example $L_j^0 W(x, y, s) = 0\ \forall j$ and $f_j(x,y,0) = \frac{\partial \psi}{\partial \overline{z}_j}(x,y)$ for some real analytic $\psi(x,y)$. We thus assume that

$$p_k(x,y,s,\xi,\eta,\sigma) = \frac{1}{2}(\xi_k + i\eta_k) - a_k\sigma^m - i\frac{\partial \phi}{\partial \overline{z}_k}(x,y)\sigma + f_k(x,y,s)$$

where the f_j are as in (5), with $f_j(x,y,0) = \frac{\partial \psi}{\partial \overline{z}_j}(x,y)$, and $L_j^0 W = 0$ for all j. Let u be a solution of the system

$$p_k(x,y,s,u_x,u_y,u_s) = 0,\ \ k = 1, \ldots, n,$$

that is,

$$\frac{\partial u}{\partial \overline{z}_k} - i\frac{\partial \phi}{\partial \overline{z}_k}\frac{\partial u}{\partial s} = a_k\left(\frac{\partial u}{\partial s}\right)^m + f_k.$$

The linearized operators are

$$L_k^u = \frac{\partial}{\partial \overline{z}_k} - \left(i\frac{\partial \phi}{\partial \overline{z}_k} + ma_k\left(\frac{\partial u}{\partial s}\right)^{m-1}\right)\frac{\partial}{\partial s},\ \ 1 \leq k \leq n.$$

Assume that $\frac{\partial \phi}{\partial \overline{z}_k}(0) = 0\ \forall k$. Observe that at the origin, the characteristic set of the linearized operators is given by

$$S = \{(\xi,\eta,\sigma) : \xi_k = 2m\sigma\Re(a_k(u_s(0))^{m-1}),\ \eta_k = 2m\sigma\Im(a_k(u_s(0))^{m-1})\}.$$

Suppose now $a_1 = a_2 = 0$, ϕ is real valued, $\phi_{\overline{z}_1 z_1}(0) > 0$ and $\phi_{\overline{z}_2 z_2}(0) < 0$. Then any C^2 solution u of the nonlinear system is real analytic near the origin. To see this, first note that by Corollary 2.3, at the origin, $WF_a u \subset S$. Next since S is one dimensional, by applying Corollary 2.5 to the vector fields L_1^u and L_2^u, we conclude that $WF_a u$ is empty at the origin and so u is real analytic in a neighborhood of the origin.

Example 2.10. Let $u(x,t)$ be a C^3 function in a neighborhood of the origin in $\mathbb{R}^2$ that satisfies the equation

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = f(x,t)$$

where $f(x,t)$ is a real analytic function. We have

$$\begin{aligned}[L^u, \overline{L^u}] &= 2(\overline{uu_t} - uu_t + \overline{uu_x}u^2 - uu_x\overline{u}^2)\frac{\partial}{\partial x} \\ &= -4i(\Im(fu - u^3u_x) + \Im(\overline{u}^2 uu_x))\frac{\partial}{\partial x}.\end{aligned}$$

Suppose now $u(0,0) = 0$. Then $[L^u, \overline{L^u}](0,0) = 0$ and

$$[L^u, [L^u, \overline{L^u}]](0,0) = -4i\Im(f(0)^2)\frac{\partial}{\partial x}.$$

Therefore, if $u(0,0) = 0$ and $\Im(f(0)^2) \neq 0$, then by Corollary 2.8, u is real analytic at the origin.
Assume next that u is a C^2 solution in the region $t \geq 0$, $u(0,0) \neq 0$, and both $u(0,0)$ and $u_x(0,0)$ are real. Then $[L^u, \overline{L^u}](0,0) = -4iu(0,0)\Im f(0,0)\frac{\partial}{\partial x}$. Hence if $\Im f(0,0) \neq 0$, by Theorem 2.4, $u(x,0)$ is microlocally real analytic either at $(0;1)$ or at $(0;-1)$.

3. Some lemmas and the proof of Theorem 2.1

Lemmas 3.2 and 3.4 are stated in [T1] without proofs. Lemma 3.1 was used in [BGT] and [HT]. In the following lemma, for a C^1 function $F(x,\zeta_0,\zeta)$ that is holomorphic in (ζ_0,ζ), H_F° will denote the principal part of the Hamiltonian H_F defined in (1.4).

Given such a function F and a C^1 function $u(x)$, F^u will denote the function given by

$$F^u(x) = F(x, u(x), u_x(x)).$$

Lemma 3.1. *Let u be a C^2 function on an open set $U \subseteq \mathcal{M}$ satisfying the system of equations*

$$F_j(x, u(x), u_x(x)) = 0 \quad \forall\, j = 1, \ldots, n$$

where the $F_j(x,\zeta_0,\zeta)$ are C^∞, holomorphic in (ζ_0,ζ). Let L_j^u be the linearized operators of $F_j(x,u(x),u_x(x)) = 0$ at u. If $G(x,\zeta_0,\zeta)$ is a C^1 function, holomorphic in (ζ_0,ζ), then

$$L_j^u(G^u) = \left(H_{F_j}^\circ G\right)^u \quad \forall\, j = 1, \ldots, n.$$

Proof. Consider the C^1 mapping

$$\Psi(x) = (x, u(x), u_x(x))$$

that maps U into (x, ζ_0, ζ) space. It is easy to see that the push forwards $\Psi_*(L_j^u)$ agree with $H^\circ_{F_j}$ on the class of C^1 functions that are holomorphic in (ζ_0, ζ). The lemma follows. □

Lemma 3.2. *The holomorphic Poisson bracket satisfies the Jacobi identity. That is,*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Proof. Observe first that the sum $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$ is a first-order differential operator in each function since for example it equals $H_f(\{g, h\})+ [H_g, H_h](f)$. Therefore, in the following computations, the notation $p \sim q$ will mean that $p - q$ is a sum of terms each of which has a second-order derivative of p or q. We will use the notation

$$[F, G] = \{F, G\} - F_{\zeta_0} G + F G_{\zeta_0}.$$

We have

$$\begin{aligned}\{f, \{g, h\}\} = &\sum_{i=1}^{N} f_{\zeta_i}(\{g, h\}_{x_i} + \zeta_i\{g, h\}_{\zeta_0}) - \sum_{i=1}^{N}\{g, h\}_{\zeta_i}(f_{x_i} + \zeta_i f_{\zeta_0})\\ &+ f_{\zeta_0}\{g, h\} - f\{g, h\}_{\zeta_0}.\end{aligned}$$

$$\{g, h\}_{x_i} = [g, h]_{x_i} + (g_{\zeta_0} h - g h_{\zeta_0})_{x_i} \sim g_{\zeta_0} h_{x_i} - g_{x_i} h_{\zeta_0}.$$

$$\begin{aligned}\{g, h\}_{\zeta_0} &= [g, h]_{\zeta_0} + (g_{\zeta_0} h - g h_{\zeta_0})_{\zeta_0} \sim [g, h]_{\zeta_0}\\ &\sim 0.\end{aligned}$$

$$\begin{aligned}\{g, h\}_{\zeta_i} &= [g, h]_{\zeta_i} + (g_{\zeta_0} h - g h_{\zeta_0})_{\zeta_i}\\ &\sim g_{\zeta_i} h_{\zeta_0} - g_{\zeta_0} h_{\zeta_i} + g_{\zeta_0} h_{\zeta_i} - g_{\zeta_i} h_{\zeta_0} \sim 0.\end{aligned}$$

It follows that

$$\begin{aligned}\{f, \{g, h\}\} &\sim \sum_{i=1}^{N} f_{\zeta_i}(g_{\zeta_0} h_{x_i} - g_{x_i} h_{\zeta_0}) + f_{\zeta_0}\{g, h\}\\ &= \sum_{i=1}^{N} f_{\zeta_i}(g_{\zeta_0} h_{x_i} - g_{x_i} h_{\zeta_0}) + \sum_{i=1}^{N} f_{\zeta_0} g_{\zeta_i}(h_{x_i} + \zeta_i h_{\zeta_0})\\ &\quad - \sum_{i=1}^{N} f_{\zeta_0} h_{\zeta_i}(g_{x_i} + \zeta_i g_{\zeta_0}) + f_{\zeta_0}(g_{\zeta_0} h - g h_{\zeta_0}),\end{aligned}$$

$$\begin{aligned}\{g,\{h,f\}\} &\sim \sum_{i=1}^{N} g_{\zeta_i}(h_{\zeta_0}f_{x_i} - h_{x_i}f_{\zeta_0}) + g_{\zeta_0}\{h,f\}\\ &= \sum_{i=1}^{N} g_{\zeta_i}(h_{\zeta_0}f_{x_i} - h_{x_i}f_{\zeta_0}) + \sum_{i=1}^{N} g_{\zeta_0}h_{\zeta_i}(f_{x_i} + \zeta_i f_{\zeta_0})\\ &\quad - \sum_{i=1}^{N} g_{\zeta_0}f_{\zeta_i}(h_{x_i} + \zeta_i h_{\zeta_0}) + g_{\zeta_0}(h_{\zeta_0}f - hf_{\zeta_0}),\end{aligned}$$

and

$$\begin{aligned}\{h,\{f,g\}\} &\sim \sum_{i=1}^{N} h_{\zeta_i}(f_{\zeta_0}g_{x_i} - f_{x_i}g_{\zeta_0}) + h_{\zeta_0}\{f,g\}\\ &= \sum_{i=1}^{N} h_{\zeta_i}(f_{\zeta_0}g_{x_i} - f_{x_i}g_{\zeta_0}) + \sum_{i=1}^{N} h_{\zeta_0}f_{\zeta_i}(g_{x_i} + \zeta_i g_{\zeta_0})\\ &\quad - \sum_{i=1}^{N} h_{\zeta_0}g_{\zeta_i}(f_{x_i} + \zeta_i f_{\zeta_0}) + h_{\zeta_0}(f_{\zeta_0}g - fg_{\zeta_0}).\end{aligned}$$

Hence the Jacobi identity holds. □

Lemma 3.3. *Let u be a C^2 function on an open set $U \subseteq \mathcal{M}$ satisfying the system of equations*

$$F_j(x,u(x),u_x(x)) = 0 \quad \forall\ j = 1,\ldots,n$$

where the $F_j(x,\zeta_0,\zeta)$ are C^∞, holomorphic in (ζ_0,ζ). Assume that the F_j satisfy (1.2) *and* (1.6)*. Then there are smooth functions $a^l_{jk}(x,\zeta_0,\zeta)$, holomorphic in (ζ_0,ζ) on the set $\Sigma = \{(x,\zeta_0,\zeta) : F_j(x,\zeta_0,\zeta) = 0,\ 1 \le j \le n\}$, such that*

$$[H_{F_j}, H_{F_k}] = \sum_{l=1}^{n} a^l_{jk}(x,\zeta_0,\zeta)H_{F_l}.$$

Proof. By Lemma 3.2, $[H_{F_j}, H_{F_k}] = H_{\{F_j,F_k\}}$ and by hypothesis, $\{F_j, F_k\} = 0$ on Σ. The lemma follows from these equations. □

Lemma 3.4. *Let u and the F_j be as in Lemma* 3.3*. Let $\mathcal{V}^u$ be the bundle generated by $L^u_1,\ldots,L^u_n$. Then $\mathcal{V}^u$ is involutive.*

Proof. First observe that on the set Σ, Lemma 3.3 implies for the principal parts $H^\circ_{F_j}$ that

$$[H^\circ_{F_i}, H^\circ_{F_j}] = \sum_{l=1}^{n} a^l_{ij}H^\circ_{F_l} \tag{3.1}$$

For each $i,j = 1,\ldots,n$, write

$$[L^u_i, L^u_j] = \sum_{r=1}^{m} c^r_{ij}(x)\frac{\partial}{\partial x_r}.$$

Since the coefficients of the Hamiltonians H_{F_i} are holomorphic in (ζ_0, ζ), by Lemma 3.1 we have:

$$L_i^u\left(L_j^u(x_r)\right) = \left[H_{F_i}^\circ\left(H_{F_j}^\circ(x_r)\right)\right]^u$$

and so

$$\begin{aligned} c_{ij}^r(x) = \left[L_i^u, L_j^u\right](x_r) &= \left(\left[H_{F_i}^\circ, H_{F_j}^\circ\right](x_r)\right)^u \\ &= \left(\sum_{l=1}^n a_{ij}^l H_{F_l}^\circ(x_r)\right)^u \qquad \text{(by (3.1))} \\ &= \sum_{l=1}^n a_{ij}^l(x, u(x), u_x(x)) L_l^u(x_r). \end{aligned}$$

Hence $[L_i^u, L_j^u]$ is a linear combination of the L_k^u. □

Lemma 3.5. *Let Σ, u, and E be as in Theorem* 2.1. *Then near each point $p \in E$, we can get real analytic coordinates (x,t), $x = (x_1, \dots, x_m)$, $t = (t_1, \dots, t_n)$ vanishing at p such that in the new coordinates,*

$$u_{t_j} = f_j(x, t, u, u_x), \quad 1 \le j \le n$$

and $E = \{(x, 0)\}$ near the origin. The f_j are real analytic in (x, t, ζ_0, ζ), and holomorphic in (ζ_0, ζ) in a neighborhood of the origin in $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{C} \times \mathbb{C}^m$.

Proof. Let $p \in E$. Let U be the domain of local coordinates $x_1, \dots, x_N$ that vanish at p. We may assume that we have n real analytic functions $F_1(x, \zeta_0, \zeta), \dots, F_n(x, \zeta_0, \zeta)$ that are holomorphic in (ζ_0, ζ), such that (1.2) and (1.6) hold and

$$F_j(x, u(x), u_x(x)) = 0, \quad 1 \le j \le n. \tag{3.2}$$

Condition (1.2) allows us to apply the implicit function theorem to (3.2) to get new functions (after relabelling coordinates)

$$F_j(x, \zeta) = \zeta_j - f_j(x, \zeta_0, \zeta_{n+1}, \dots, \zeta_N), \quad 1 \le j \le n$$

where the f_j are real analytic and holomorphic in (ζ_0, ζ).

Write $t_j = x_j$ for $1 \le j \le n$, $\tau_j = \zeta_j$ for $1 \le j \le n$, x_i instead of x_{n+i} and ζ_i instead of ζ_{n+i} for $n < i \le m = N - n$. With this notation, u satisfies:

$$u_{t_j} = f_j(x, t, u, u_x), \quad 1 \le j \le n.$$

The linearized bundle $\mathcal{V}^u$ is generated by

$$L_j^u = \frac{\partial}{\partial t_j} - \sum_{l=1}^m \frac{\partial f_j}{\partial \zeta_l}(x, t, u, u_x)\frac{\partial}{\partial x_l}, \quad 1 \le j \le n.$$

Near the origin, we may assume that E has the form

$$E = \{(x', g(x', t'), t', h(x', t'))\},$$

where g and h are real analytic, $x = (x', x'')$, $t = (t', t'')$, $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{m-k}$, $t' \in \mathbb{R}^r$, $t'' \in \mathbb{R}^{n-r}$ and $k + r = m$. Change coordinates to $y = (y', y'')$ and

$s = (s', s'')$ where $y' = x'$, $y'' = x'' - g(x', t')$, $s' = t'$ and $s'' = t'' - h(x', t')$. In these new coordinates, $E = \{(y', 0, s', 0)\}$, and if $\eta = (\eta', \eta'')$ and $\sigma = (\sigma', \sigma'')$ denote the duals of $y = (y', y'')$ and $s = (s', s'')$, u satisfies equations of the form

$$u_{s_j} = p_j(y, s, u, u_y, u_{s''}), \quad 1 \leq j \leq n \tag{3.3}$$

where the $p_j(y, s, \eta_0, \eta, \sigma'')$ are real analytic and holomorphic in (η, σ). Let

$$F_j(y, s, \eta_0, \eta, \sigma) = \sigma_j - p_j(y, s, \eta_0, \eta, \sigma''), \quad 1 \leq j \leq n.$$

Since the linearized bundle $\mathcal{V}^u$ of (3.3) is transversal to $E = \{(y', 0, s', 0)\}$, the Jacobian

$$\frac{\partial(F_1, \ldots, F_n)}{\partial(\eta'', \sigma'')}$$

is invertible. By the implicit function theorem, the equations

$$F_j(y, s, \eta_0, \eta, \sigma) = 0 \quad 1 \leq j \leq n$$

lead to solutions

$$\eta_j = f_j(y, s, \eta_0, \eta', \sigma'), \quad k + 1 \leq j \leq m$$

and

$$\sigma_l = f_l(y, s, \eta_0, \eta', \sigma'), \quad r + 1 \leq l \leq n$$

where the f_i are real analytic, holomorphic in (η_0, η, σ). We can now relabel the coordinates and their duals and assume that we have coordinates (x, t), $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, with dual coordinates (ζ, τ) such that near the origin, $E = \{(x, 0)\}$, and u satisfies

$$u_{t_j} = f_j(x, t, u, u_x), \quad 1 \leq j \leq n$$

with f_j real analytic, holomorphic in (ζ_0, ζ). □

Proof of Theorem 2.1. Using Lemma 3.5, we may assume we have coordinates (x, t) near $0 \in E$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, $E = \{(x, 0)\}$ near 0 and on the wedge $\mathcal{W}$,

$$u_{t_j} = f_j(x, t, u, u_x), \quad 1 \leq j \leq n \tag{3.4}$$

Let

$$L_j^u = \frac{\partial}{\partial t_j} - \sum_{k=1}^{m} \frac{\partial f_j}{\partial \zeta_k}(x, t, u, u_x) \frac{\partial}{\partial x_k}.$$

Note that the L_j^u are C^1 on $\overline{\mathcal{W}}$. Consider the principal parts of the holomorphic Hamiltonians

$$H_j^\circ = \frac{\partial}{\partial t_j} - \sum_{k=1}^{m} \frac{\partial f_j}{\partial \zeta_k}(x, t, \zeta_0, \zeta) \frac{\partial}{\partial x_k} + \sum_{l=1}^{m} \left(\frac{\partial f_j}{\partial t_l} + \tau_l \frac{\partial f_j}{\partial \zeta_0} \right) \frac{\partial}{\partial \tau_l}$$
$$+ \sum_{i=1}^{m} \left(\frac{\partial f_j}{\partial x_i} + \zeta_i \frac{\partial f_j}{\partial \zeta_0} \right) \frac{\partial}{\partial \zeta_i} + \left(- \sum_{i=1}^{m} \zeta_i \frac{\partial f_j}{\partial \zeta_i} + f_j \right) \frac{\partial}{\partial \zeta_0}.$$

Since the system (3.4) is involutive, there exist real analytic functions

$$Z_1(x, t, \zeta_0, \zeta), \ldots, Z_m(x, t, \zeta_0, \zeta), \quad W_0(x, t, \zeta_0, \zeta), \ldots, W_m(x, t, \zeta_0, \zeta),$$

holomorphic in (ζ_0, ζ) and satisfying

$$H_j^0 Z_k = 0, \quad Z_k(x, 0, \zeta_0, \zeta) = x_k, \quad H_j^0 W_l = 0, \quad W_l(x, 0, \zeta_0, \zeta) = \zeta_l \tag{3.5}$$

for $1 \le k \le m$, $0 \le l \le m$, near $(x, t) = 0$ in $\mathbb{R}^{m+n}$ and $(\zeta_0, \zeta) = 0$ in $\mathbb{C}^{m+1}$.

Let

$$Z_k^u(x,t) = Z_k(x, t, u(x,t), u_x(x,t)), \text{ and } W_l^u(x,t) = W_l(x, t, u(x,t), u_x(x,t)).$$

By Lemma 3.1 and (3.5),

$$L_j^u(Z_k^u) = 0 \quad \text{and} \quad L_j^u(W_l^u) = 0$$

in the wedge $\mathcal{W}$. In particular, since $Z_k^u(x, 0) = x_k$, the bundle $\mathcal{V}^u$ is locally integrable in the wedge $\mathcal{W}$ near 0. We have

$$W_0^u(x,0) = u(x,0) = u_0(x) \text{ and } W_j^u(x,0) = u_{0x_j}(x) \tag{3.6}$$

for $x \in E$.

Let $\sigma \in T_0^*(E)$ such that $\sigma \notin (\Gamma_0^T(\mathcal{W}))^0$. Then there exists $L_0 \in \mathcal{V}_0^u$ with $\Im L_0 \in \Gamma_0(\mathcal{W})$, $\Re L_0 \in T_0 E$ such that $\langle \sigma, \Re L_0 \rangle < 0$. Write

$$L_0 = \sum_{j=1}^m a_j \frac{\partial}{\partial x_j} + i \left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i} + \sum_{l=1}^n c_l \frac{\partial}{\partial t_l} \right)$$

where the a_j, b_i and $c_l \in \mathbb{R}$. Since $\Im L_0 \notin T_0 E$, $c_l \ne 0$ for some l. Let

$$Y = \{(x, c_1 s, \ldots, c_n s) : \ x \in E, 0 < s < \delta\}$$

for some small $\delta > 0$. Then Y is a wedge in an $m+1$-dimensional space with edge E and $Y \subseteq \mathcal{W}$.

Y inherits a locally integrable structure from $\mathcal{V}^u$ generated by a vector field L that is C^1 up to E and $L(0) = L_0$. The restriction of $W_0^u(x, y)$ to Y is a solution of L. We can then proceed as in the proof of Theorem III.1 in [EG] to conclude that $\sigma \notin WF_a(W_0^u(x, 0))$, that is, $\sigma \notin WF_a(u_0)$. □

We will next derive an edge of the wedge result as a consequence of Corollary 2.2 using ideas from the proof of Theorem X.3.1 in [T1].

Corollary 3.6. *Let p, Σ, $\mathcal{W}^+$, $\mathcal{W}^-$, E, u^+, u^-, u_0 be as in Corollary 2.2. Assume that $\mathcal{V}_p = \mathcal{V}_p^{u^+} = \mathcal{V}_p^{u^-}$ is elliptic, that is, $T_p^\circ(\mathcal{M}) = \emptyset$. Then there exists a real analytic function u in a neighborhood of p that is a solution and agrees with u^+ on $\mathcal{W}^+$ and with u^- on $\mathcal{W}^-$.*

Proof. We may assume that we are in the coordinates (x, t) where $E = \{(x, 0)\}$ and p is 0,

$$u_{t_j}^+ = f_j(x, t, u^+, u_x^+) \text{ in } \mathcal{W}^+, \text{ and } u_{t_j}^- = f_j(x, t, u^-, u_x^-) \text{ in } \mathcal{W}^-, \quad 1 \le j \le n.$$

We also have

$$u^+(x, 0) = u_0(x) = u^-(x, 0).$$

By hypothesis and Corollary 2.2, $u_0(x)$ is a real analytic function. Consider the map

$$F(z,w,\zeta_0,\zeta) = (Z(z,w,\zeta_0,\zeta), w, W(z,w,\zeta_0,\zeta))$$

which is holomorphic near $(0,0,u_0(0),u_{0x}(x))$ and

$$F(0,0,u_0(0),u_{0x}(0)) = (0,0,u_0(0),u_{0x}(x)).$$

Here $Z = (Z_1,\ldots,Z_m)$ and $W = (W_0,\ldots,W_m)$ are as in (3.5). Let

$$G(z',w',\zeta_0',\zeta') = (P(\zeta',w',\zeta_0',\zeta'), w', Q(\zeta',w',\zeta_0',\zeta'))$$

denote the inverse of F. Then Q is holomorphic and

$$(\zeta_0,\zeta) = Q(Z(z,w,\zeta_0,\zeta), w, W(z,w,\zeta_0,\zeta)).$$

In particular,

$$u^+(x,t) = Q_0(Z^{u^+}(x,t),t,W^{u^+}(x,t)),\ u^+_{x_j}(x,t) = Q_j(Z^{u^+}(x,t),t,W^{u^+}(x,t)) \tag{3.7}$$

for $1 \le j \le m$, and similarly,

$$u^-(x,t) = Q_0(Z^{u^-}(x,t),t,W^{u^-}(x,t)),\ u^-_{x_j}(x,t) = Q_j(Z^{u^-}(x,t),t,W^{u^-}(x,t)) \tag{3.8}$$

for $1 \le j \le m$. Since u_0 is real analytic, the function $u_0(Z^{u+}(x,t))$ is a solution of $\mathcal{V}^{u^+}$ in $\mathcal{W}^+$. Recall that $W_0^{u^+}(x,t)$ is also a solution of $\mathcal{V}^{u^+}$ in $\mathcal{W}^+$ and $u_0(Z^{u^+}(x,0)) = u_0(x) = W_0^{u^+}(x,0)$. By the wedge version of the Baouendi-Treves approximation theorem (see [HM], [BCH]), it follows that on $\mathcal{W}^+$,

$$W_0^{u^+}(x,t) = u_0(Z^{u^+}(x,t)),$$

and likewise on $\mathcal{W}^-$,

$$W_0^{u^-}(x,t) = u_0(Z^{u^-}(x,t)).$$

We also have

$$W_j^{u^+}(x,t) = u_{0x_j}(Z^{u^+}(x,t)) \text{ on } \mathcal{W}^+, \text{ and}$$

$$W_j^{u^-}(x,t) = u_{0x_j}(Z^{u^-}(x,t)) \text{ on } \mathcal{W}^-, \text{ for } 1 \le j \le m.$$

Going back to (3.7), we get:

$$u^+(x,t) = Q_0(Z^{u^+}(x,t),t,u_0(Z^{u^+}(x,t)),u_{0x}(Z^{u^+}(x,t))) \tag{3.9}$$

$$u^+_{x_i}(x,t) = Q_i(Z^{u^+}(x,t),t,u_0(Z^{u^+}(x,t)),u_{0x}(Z^{u^+}(x,t))),$$

for $1 \le i \le m$ on $\mathcal{W}^+$. Next, since

$$\frac{\partial Z}{\partial \zeta_j}(0,0,u_0(0),u_x(0)) = 0 \ \forall\ j = 0,\ldots,m,$$

we can apply the implicit function theorem to the system

$$\zeta_0 = Q_0(Z(x,t,\zeta_0,\zeta),t,u_0(Z(x,t,\zeta_0,\zeta)),u_{0x}(Z(x,t,\zeta_0,\zeta)))$$

$$\zeta_i = Q_i(Z(x,t,\zeta_0,\zeta),t,u_0(Z(x,t,\zeta_0,\zeta)),u_{0x}(Z(x,t,\zeta_0,\zeta))),\ 1 \le i \le m$$

to find real analytic function $\zeta_0 = v_0(x,t)$, $\zeta = v(x,t)$ such that

$$v_0(x,t) = Q_0(Z(x,t,v_0,v), u_0(Z(x,t,v_0,v)), u_{0x}(Z(x,t,v_0,v)))$$

and

$$v_i(x,t) = Q_i(Z(x,t,v_0,v), u_0(Z(x,t,v_0,v)), u_{0x}(Z(x,t,v_0,v)))$$

for $1 \le i \le m$. By uniqueness and (3.9), we conclude that $v_0(x,t) = u^+(x,t)$ in $\mathcal{W}^+$. Likewise, $v_0(x,t) = u^-(x,t)$ in $\mathcal{W}^-$. Set $u(x,t) = v_0(x,t)$. Since $u_{t_j} = f_j(x,t,u,u_x)$ on $\mathcal{W}^+$ for $1 \le j \le n$, by analyticity, it follows that

$$u_{t_j} = f_j(x,t,u,u_x) \text{ for all } j$$

in a full neighborhood of the origin in $\mathbb{R}^N$ $(N = m+n)$. □

4. Proofs of Theorem 2.4 and Theorem 2.7

To prove Theorem 2.4, we will use the coordinates and notations of the proof of Theorem 2.1. In particular, $E = \{(x,0)\}$ near our central point $p = 0$. The function u is C^2 on $\overline{\mathcal{W}}$ and solves

$$u_{t_j} = f_j(x,t,u,u_x) \text{ on } \mathcal{W}, \quad 1 \le j \le n.$$

We are given a C^1 section L of $\mathcal{V}_0^u$ on $\overline{\mathcal{W}}$ such that $L_0 = L(0) \in \Gamma_0^{\mathcal{V}}(\mathcal{W})$, $\sigma \in T_0^\circ\mathcal{M}$, and $\frac{1}{i}\langle\sigma, [L, \overline{L}]\rangle < 0$.

We can write

$$L_0 = \sum_{j=1}^{m} a_j \frac{\partial}{\partial x_j} + i\left(\sum_{k=1}^{m} b_k \frac{\partial}{\partial x_k} + \sum_{l=1}^{n} c_l \frac{\partial}{\partial t_l}\right)$$

where the a_j, b_k, and $c_l \in \mathbb{R}$. We also know that $c_l \neq 0$ for some l. As before, let

$$Y = \{(x, c_1 s, \ldots, c_n s) \;:\; x \in E,\ 0 < s < \delta\}$$

for some small $\delta > 0$ so that $Y \subseteq \mathcal{W}$. Assume without loss of generality that $c_1 \neq 0$. After a linear change of coordinates that preserves E we may assume that

$$Y = \{(x, t, 0, \ldots, 0) \;:\; x \in E,\ 0 < t < \delta\}.$$

Y inherits a locally integrable structure generated by a C^1 vector field which we still denote by L such that $L_0 \in \Gamma_0^{\mathcal{V}}(\mathcal{W})$ and $\frac{1}{i}\langle\sigma, [L, \overline{L}]\rangle < 0$. The C^2 solution u restricts to a C^2 function $u(x,t)$ on Y. Recall next that $Z_j(x, 0, \zeta_0, \zeta) = x_j$ for $1 \le j \le m$.

It follows that

$$Z_j(x,t) = Z_j(x,t,u(x,t),u_x(x,t))$$

have the form

$$Z_j(x,t) = x_j + t\Psi_j(x,t)$$

where $\Psi_1(x,t), \ldots, \Psi_m(x,t)$ are C^1 functions on $\overline{Y}$. Since $L_0 \notin T_0E$, we may assume that L has the form

$$L = i\frac{\partial}{\partial t} + \sum_{j=1}^{m} c_j(x,t)\frac{\partial}{\partial x_j} \tag{4.1}$$

where the c_j are C^1 on $\overline{Y}$.

Observe that at a point $(x,0)$ near the origin, the characteristic set of L is given by

$$\mathrm{Char}\, L|_{(x,0)} = \{(x,0;\xi,\tau) : \Im\Psi(x,0)\cdot\xi = 0,\ \tau = \Re\Psi(x,0)\cdot\xi,\ (\xi,\tau) \neq (0,0)\}. \tag{4.2}$$

The latter follows from the equations,

$$c(x,t) = -iZ_x^{-1}\cdot Z_t,\ Z_x = I + t\Psi_x, \quad \text{and } Z_t = \Psi + t\Psi_t.$$

The proof of Theorem 2.4 now follows from the following lemma from [B].

Lemma 4.1. *Suppose L as above has C^1 coefficients and the $\Psi_j \in C^1(\overline{Y})$. Let $h \in C^1(\overline{Y})$ be a solution of $Lh = 0$. If $\sigma = (0,0;\xi^0,\tau^0) \in \mathrm{Char}\, L$ and $\frac{1}{2i}\sigma([L,\overline{L}]) < 0$, then $(0,\xi^0) \notin WF_a h(x,0)$.*

Proof. By adding a variable we may assume that L is a CR vector field near the origin. Indeed, we can add the variable s and consider the vector field

$$L' = L + \frac{\partial}{\partial s}$$

which is locally integrable with first integrals

$$Z_1(x,t), \ldots, Z_m(x,t), \quad \text{and} \quad Z_{m+1}(x,t,s) = s + it.$$

Note that $L'h = 0$ for $t > 0$ and L' is a CR vector field. We can thus assume that our original L is CR. This means that for some j, $\Im\Psi_j(0) \neq 0$. Without loss of generality assume that

$$\Im\Psi_1(0) \neq 0. \tag{4.3}$$

Observe next that the linear change of coordinates

$$x_l' = x_l + t\Re\Psi_l(0),\ t' = t$$

allow us to assume, after dropping the primes, that

$$\Re\Psi_j(0) = 0,\ \text{for all } j = 1,\ldots,m. \tag{4.4}$$

We can use (4.3) and (4.4) to replace $Z_2, \ldots, Z_m$ by a linear combination of $Z_1, \ldots, Z_m$ and apply a linear change of coordinates to get

$$Z_j = x_j + t\Psi_j,\ 1 \leq j \leq m,\ \text{and } \Psi_1(0) = i, \Psi_j(0) = 0,\ \text{for } 2 \leq j \leq m. \tag{4.5}$$

The equation $LZ_l = 0$ implies that

$$i\left(\Psi_l + t\frac{\partial\Psi_l}{\partial t}\right) + c_l + \sum_{j=1}^{m} c_j t\frac{\partial\Psi_l}{\partial x_j} = 0 \tag{4.6}$$

and so from (4.5) and (4.6),

$$c_1(0) = 1 \text{ and } c_j(0) = 0 \text{ for } j \geq 2. \tag{4.7}$$

The condition that $(0, 0; \xi^0, \tau^0) \in \operatorname{Char} L$ therefore means that $\tau^0 = 0 = \xi_1^0$ and $\xi_j^0 \neq 0$ for some $j \geq 2$. In particular, $\xi^0 \neq 0$ and

$$\xi^0 \cdot \Im\Psi(0) = 0. \tag{4.8}$$

We may assume that

$$\xi^0 = (0, 1, 0, \ldots, 0). \tag{4.9}$$

We have

$$[L, \overline{L}] = \sum_{l=1}^{m} A_l(x,t) \frac{\partial}{\partial x_l}$$

where

$$A_l(x,t) = i\left(\frac{\partial \overline{c_l}}{\partial t} + \frac{\partial c_l}{\partial t}\right) + \sum_{j=1}^{m} c_j \frac{\partial \overline{c_l}}{\partial x_j} - \sum_{j=1}^{m} \overline{c_j} \frac{\partial c_l}{\partial x_j}. \tag{4.10}$$

We will express $A_l(0,0)$ using the Ψ_j. From (4.6) we have

$$i\Psi_l(x,0) + c_l(x,0) = 0. \tag{4.11}$$

Subtract (4.11) from (4.6), divide by t, and let $t \to 0$ to arrive at (recalling that Ψ and L are C^1):

$$2i\frac{\partial \Psi_l}{\partial t}(x,0) + \frac{\partial c_l}{\partial t}(x,0) + \sum_{j=1}^{m} c_j(x,0) \frac{\partial \Psi_l}{\partial x_j}(x,0) = 0. \tag{4.12}$$

From (4.11) and (4.12), we get:

$$\frac{\partial c_l}{\partial t}(x,0) = -2i\frac{\partial \Psi_l}{\partial t}(x,0) + i\sum_{j=1}^{m} \Psi_j(x,0) \frac{\partial \Psi_l}{\partial x_j}(x,0). \tag{4.13}$$

Thus from (4.5), (4.10), (4.11) and (4.13), we have

$$\begin{aligned} A_l(0,0) &= i\left(\frac{\partial \overline{c_l}}{\partial t}(0,0) + \frac{\partial c_l}{\partial t}(0,0)\right) + \frac{\partial \overline{c_l}}{\partial x_1}(0,0) - \frac{\partial c_l}{\partial x_1}(0,0) \\ &= -2\left(\frac{\partial \overline{\Psi_l}}{\partial t}(0,0) - \frac{\partial \Psi_l}{\partial t}(0,0)\right) \\ &= 4i\frac{\partial \Im\Psi_l}{\partial t}(0). \end{aligned}$$

Therefore, the assumption that $\frac{1}{2i}\sigma([L, \overline{L}]) < 0$ implies that

$$\frac{\partial \Im\Psi_2}{\partial t}(0) = \frac{\partial \Im\Psi}{\partial t}(0) \cdot \xi^0 < 0. \tag{4.14}$$

Next, we show that coordinates (x,t) and first integrals $Z_l = x_l + t\Psi_l$ can be chosen so that (4.5), (4.9) and (4.14) still hold and in addition,

$$\frac{\partial \Im\Psi_l}{\partial x_j}(0) = 0 \quad \text{for all } l, j.$$

Define

$$\widetilde{Z}_l(x,t) = Z_l + \sum_{k=1}^{m} a_{lk} Z_1 Z_k, \quad l = 1, \dots, m,$$

where

$$a_{l,k} = \begin{cases} -\frac{1}{2}\dfrac{\partial \Im \Psi_l}{\partial x_k}(0), & k = 1 \\ -\dfrac{\partial \Im \Psi_l}{\partial x_k}(0), & 2 \le k \le m. \end{cases}$$

Note that

$$\widetilde{Z}_l(x,t) = x_l + \sum_{k=1}^{m} a_{lk} x_1 x_k + t\widetilde{\Psi_l}(x,t),$$

where

$$\widetilde{\Psi_l}(x,t) = \Psi_l + \sum_k a_{l,k}(x_1 \Psi_k + x_k \Psi_1 + t \Psi_1 \Psi_k).$$

By the choice of the a_{lk} and the fact that $\Psi_1(0) = i$, we have

$$\frac{\partial \Im \widetilde{\Psi_l}}{\partial x_j}(0) = 0 \text{ for all } l, j.$$

Introduce new coordinates

$$\tilde{x_l} = x_l + \sum_{k=1} a_{lk} x_1 x_k, \quad \tilde{t} = t, \ 1 \le l \le m.$$

These change of coordinates are smooth and hence L is still C^1 in these coordinates. After dropping the tildes both in the new coordinates and the first integrals, we have:

$$Z_j = x_j + t\Psi_j \text{ with } \frac{\partial \Im \Psi_l}{\partial x_j}(0) = 0 \text{ for all } l, j \tag{4.15}$$

and (4.5), (4.9) and (4.14) still hold. Moreover, the new coordinates preserve the set $\{t = 0\}$ and so L still has the form

$$L = i\frac{\partial}{\partial t} + \sum_{j=1}^{m} c_j(x,t)\frac{\partial}{\partial x_j}.$$

Let $\eta(x) \in C_0^\infty(B_r(0))$, where $B_r(0)$ is a ball of small radius r centered at $0 \in \mathbb{R}^m$ and $\eta(x) \equiv 1$ when $|x| \le r/2$. We will be using the FBI transform

$$F_\kappa(t,z,\zeta) = \int_{\mathbb{R}^m} e^{i\zeta.(z - Z(x,t)) - \kappa\langle\zeta\rangle[z - Z(x,t)]^2} \eta(x) h(x,t)\, dZ$$

where for $z \in \mathbb{C}^m$, we write $[z]^2 = \sum_{j=1}^m z_j^2$, $\langle\zeta\rangle = (\zeta \cdot \zeta)^{1/2}$ is the main branch of the square root, $dZ = dZ_1 \wedge \cdots \wedge dZ_m = \det Z_x(x,t) dx_1 \wedge \cdots \wedge dx_m$, and $\kappa > 0$ is a parameter which will be chosen later.

To prove that $(0,\xi^0)\notin WF_a(h(x,0))$, we need to show that for some $\kappa>0$ and constants $C_1,C_2>0$,

$$\begin{aligned}|F_\kappa(0,z,\zeta)| &= \left|\int e^{i\zeta.(z-x)-\kappa\langle\zeta\rangle[z-x]^2}\eta(x)h(x,0)\,dx\right| \\ &\le C_1 e^{-c_2|\zeta|}\end{aligned}\tag{4.16}$$

for z near 0 in $\mathbb{C}^m$ and ζ in a conic neighborhood of ξ^0 in $\mathbb{C}^m$. Let $U=B_r(0)\times(0,\delta)$ for some δ small. Since h and the Z_j are solutions, the form

$$\omega = e^{i\zeta.(z-Z(x,t))-\kappa\langle\zeta\rangle[z-Z(x,t)]^2}h(x,t)dZ_1\wedge dZ_2\wedge\cdots\wedge dZ_m$$

is a closed form. This is well known when the Z_j are C^2 and when they are only C^1 as in our case, one can prove that ω is closed by approximating the Z_j by smoother functions. By Stokes' theorem, we therefore have

$$F_\kappa(0,z,\zeta)=\int_{\{t=0\}}\eta\omega=\int_{t=\delta}\eta\omega-\iint_U d\eta\wedge\omega.\tag{4.17}$$

We will show that κ,δ and $r>0$ can be chosen so that each of the two integrals on the right side of (4.17) satisfies an estimate of the form (4.16). Set

$$Q(z,\zeta,x,t)=\frac{\Re(i\zeta\cdot(z-Z(x,t))-\kappa\langle\zeta\rangle[z-Z(x,t)]^2)}{|\zeta|}.$$

Since Q is homogeneous of degree 0 in ζ, it is sufficient to show that there is $C>0$ so that $Q(0,\xi^0,x,t)\le -C$ for $(x,t)\in(\operatorname{supp}\eta\times\{\delta\})\cup(\operatorname{supp}d\eta\times[0,\delta])$. For then, $Q(z,\zeta,x,t)\le -C/2$ for the same (x,t), z near 0 in $\mathbb{C}^m$, and ζ in a conic neighborhood of ξ^0 in $\mathbb{C}^m$. We recall that $\xi^0=(0,1,\dots,0)$, and so $|\xi^0|=1$. We have:

$$\begin{aligned}Q(0,\xi^0,x,t)&=\Re(-i\xi^0\cdot(x+t\Psi)-\kappa[x+t\Psi]^2)\\ &=t\xi^0\cdot\Im\Psi(x,t)-\kappa[|x|^2+t^2|\Re\Psi|^2+2t\langle x,\Re\Psi\rangle-t^2|\Im\Psi|^2].\end{aligned}\tag{4.18}$$

Since Ψ is C^1, using (4.5), (4.14), and (4.15),

$$t(\xi^0\cdot\Im\Psi(x,t))=-C_1t^2+o(|x|t+t^2)\tag{4.19}$$

where $C_1=-\dfrac{\partial\Im\Psi}{\partial t}(0)\cdot\xi^0>0$. Let $C\ge|\Im\Psi|^2+1$ on U, and set $\alpha=\frac{C_1}{8C}$. Note that (4.19) allows us to choose r and δ small enough so that on U,

$$t(\xi^0\cdot\Im\Psi(x,t))\le-\frac{C_1}{2}t^2+\alpha|x|^2.\tag{4.20}$$

From (4.18) and (4.20), we get:

$$Q(0,\xi^0,x,t)\le-\frac{C_1}{2}t^2+\alpha|x|^2-\kappa[|x|^2-2t|x||\Re\Psi|-t^2|\Im\Psi|^2].$$

Since $\Re\Psi(0)=0$, we may assume r and δ are small enough so that

$$2t|x||\Re\Psi|\le t^2+|x|^2/2$$

and hence using $C \geq |\Im\Psi|^2 + 1$,

$$Q(0, \xi^0, x, t) \leq -\frac{C_1}{2}t^2 + \alpha|x|^2 - \kappa|x|^2 - \kappa|x|^2/2 + \kappa C t^2.$$

Choose $\kappa = \frac{3C_1}{8C}$. Recalling that $\alpha = \frac{C_1}{8C}$, we get:

$$Q(0, \xi^0, x, t) \leq -\frac{C_1}{8}t^2 - \frac{C_1}{16C}|x|^2$$

and so on $\operatorname{supp}\eta \times \{\delta\} \cup (\operatorname{supp}(d\eta) \times [0, \delta])$, $Q(0, \xi^0, x, t) \leq -C$ for some $C > 0$. This proves the lemma. □

To prove Theorem 2.7, once again we will use the coordinates and notations of the proof of Theorem 2.4. Thus $E = \{(x, 0)\}$ near $p = 0$, $u \in C^3(\overline{\mathcal{W}})$ and

$$u_{t_j} = f_j(x, t, u, u_x) \quad \text{on } \mathcal{W},\ 1 \leq j \leq n.$$

We may assume that for some $\delta > 0$,

$$Y = \{(x, t, 0, \dots, 0) : x \in E,\ 0 < t < \delta\} \subseteq \mathcal{W},$$

and L is a C^2 vector field on $\overline{Y}$ of the form

$$L = \sum_{j=1}^{m} a_j(x, t)\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial t}$$

such that $\langle \sigma, L \rangle = 0$, $\langle \sigma, [L, \bar{L}] \rangle = 0$, and

$$\sqrt{3}\,|\Im\langle \sigma, [L, [L, \bar{L}]]\rangle| < \Re\langle \sigma, [L, [L, \bar{L}]]\rangle.$$

Since u is C^3 on $\overline{\mathcal{W}}$, the functions $\psi_j(x, t)$ are C^2 on $\overline{Y}$. The rest of the proof proceeds as in the proof of Lemma 3.2 in [B] starting from (3.27) in that paper. Note in particular that unlike the reasoning used in [B], L does not have to satisfy the solvability of the Cauchy problems (3.22) in [B]. This approach therefore is simpler than the one employed in [B].

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Gevrey Hypoellipticity for an Interesting Variant of Kohn's Operator

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Dedicated to Professor Linda Preiss Rothschild on the occasion of her 60th birthday

Abstract. In this paper we consider the analogue of Kohn's operator but with a point singularity,

$$P = BB^* + B^*(t^{2\ell} + x^{2k})B, \qquad B = D_x + ix^{q-1}D_t.$$

We show that this operator is hypoelliptic and Gevrey hypoelliptic in a certain range, namely $k < \ell q$, with Gevrey index $\frac{\ell q}{\ell q - k} = 1 + \frac{k}{\ell q - k}$. Work in progress by the present authors suggests that, outside the above range of the parameters, i.e., when $k \geq \ell q$, the operator is not even hypoelliptic.

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1. Introduction

In J.J. Kohn's recent paper [11] (see also [6]) the operator

$$E_{m,k} = L_m\overline{L_m} + \overline{L_m}|z|^{2k}L_m, \quad L_m = \frac{\partial}{\partial z} - i\bar{z}|z|^{2(m-1)}\frac{\partial}{\partial t}$$

was introduced and shown to be hypoelliptic, yet to lose $2 + \frac{k-1}{m}$ derivatives in L^2 Sobolev norms. Christ [7] showed that the addition of one more variable destroyed hypoellipticity altogether.

In a recent volume, dedicated to J.J. Kohn, A. Bove and D.S. Tartakoff, [5], showed that Kohn's operator with an added Oleinik-type singularity, of the form studied in [4],

$$E_{m,k} + |z|^{2(p-1)}D_s^2$$

is s-Gevrey hypoelliptic for any $s \geq \frac{2m}{p-k}$, (here $2m > p > k$). A related result is that for the 'real' version, with $X = D_x + ix^{q-1}D_t$, where $D_x = i^{-1}\partial_x$,

$$R_{q,k} + x^{2(p-1)}D_s^2 = XX^* + (x^k X)^*(x^k X) + x^{2(p-1)}D_s^2$$

is sharply s-Gevrey hypoelliptic for any $s \geq \frac{q}{p-k}$, where $q > p > k$ and q is an even integer.

In this paper we consider the operator

$$P = BB^* + B^*(t^{2\ell} + x^{2k})B, \qquad B = D_x + ix^{q-1}D_t, \tag{1.1}$$

where k, ℓ and q are positive integers, q even.

Observe that P is a sum of three squares of complex vector fields, but, with a small change not altering the results, we could make P a sum of two squares of complex vector fields in two variables, depending on the same parameters, e.g., $BB^* + B^*(t^{2\ell} + x^{2k})^2 B$.

Let us also note that the characteristic variety of P is $\{x = 0, \xi = 0\}$, i.e., a codimension two analytic symplectic submanifold of $T^*\mathbb{R}^2 \setminus 0$, as in the case of Kohn's operator. Moreover the Poisson-Treves stratification for P has a single stratum thus coinciding with the characteristic manifold of P.

We want to analyze the hypoellipticity of P, both in C^∞ and in Gevrey classes. As we shall see the Gevrey classes play an important role. Here are our results:

Theorem 1.1. *Let P be as in* (1.1)*, q even.*

(i) *Suppose that*

$$\ell > \frac{k}{q}. \tag{1.2}$$

Then P is C^∞ hypoelliptic (in a neighborhood of the origin) with a loss of $2\frac{q-1+k}{q}$ derivatives.

(ii) *Assume that the same condition as above is satisfied by the parameters ℓ, k and q. Then P is s-Gevrey hypoelliptic for any s, with*

$$s \geq \frac{\ell q}{\ell q - k}. \tag{1.3}$$

(iii) *Assume now that*

$$\ell \leq \frac{k}{q}. \tag{1.4}$$

Then P is not C^∞ hypoelliptic.

The proof of the above theorem is lengthy, will be given in a forthcoming paper [3]. In this paper we prove items (i) and (ii) of the theorem.

It is worth noting that the operator P satisfies the complex Hörmander condition, i.e., the brackets of the fields of length up to $k+q$ generate a two-dimensional complex Lie algebra $\mathbb{C}^2$. Note that in the present case the vector fields involved are B^*, $x^k B$ and $t^\ell B$, but only the first two enter in the brackets spanning $\mathbb{C}^2$.

A couple of remarks are in order. The above theorem seems to us to suggest strongly that Treves conjecture cannot be extended to the case of sums of squares of complex vector fields, since lacking C^∞ hypoellipticity we believe that P is not analytic hypoelliptic for any choice of the parameters. We will address this point further in the subsequent paper.

The second and trivial remark is that, even in two variables, there are examples of sums of squares of complex vector fields, satisfying the Hörmander condition, that are *not* hypoelliptic. In this case the characteristic variety is a symplectic manifold. In our opinion this is due to the point singularity exhibited by the second and third vector field, or by $(t^{2\ell} + x^{2k})B$ in the two-fields version.

Restricting ourselves to the case q even is no loss of generality, since the operator (1.1) corresponding to an odd integer q is plainly hypoelliptic and actually subelliptic, meaning by that term that there is a loss of less than two derivatives. This fact is due to special circumstances, i.e., that the operator B^* has a trivial kernel in that case. We stress the fact that the Kohn's original operator, in the complex variable z, automatically has an even q, while in the "real case" the parity of q matters.

We also want to stress microlocal aspects of the theorem: the characteristic manifold of P is symplectic in $T^*\mathbb{R}^2$ of codimension 2 and as such it may be identified with $T^*\mathbb{R} \setminus 0 \sim \{(t,\tau) \mid \tau \neq 0\}$ (leaving aside the origin in the τ variable). On the other hand, the operator $P(x, t, D_x, \tau)$, thought of as a differential operator in the x-variable depending on (t, τ) as parameters, for $\tau > 0$ has an eigenvalue of the form $\tau^{2/q}(t^{2\ell} + a(t, \tau))$, modulo a non zero function of t. Here $a(t, \tau)$ denotes a (non-classical) symbol of order -1 defined for $\tau > 0$ and such that $a(0, \tau) \sim \tau^{-\frac{2k}{q}}$. Thus we may consider the pseudodifferential operator $\Lambda(t, D_t) = \operatorname{Op}\left(\tau^{2/q}(t^{2\ell} + a(t, \tau))\right)$ as defined in a microlocal neighborhood of our base point in the characteristic manifold of P. One can show that the hypoellipticity properties of P are shared by Λ, e.g., P is C^∞ hypoelliptic iff Λ is.

The last section of this paper includes a computation of the symbol of Λ as well as the proof that P is hypoelliptic if Λ is hypoelliptic. This is done following ideas of Boutet de Monvel, Helffer and Sjöstrand.

2. The operator P is C^∞ hypoelliptic

Theorem 2.1. *Under the restriction that $k < \ell q$, q even, the operator P is hypoelliptic.*

Denoting by W_j, $j = 1, 2, 3, 4$ the operators

$$W_1 = B^*, \quad W_2 = t^\ell B, \quad W_3 = x^k B, \quad \text{and} \quad W_4 = \langle D_t\rangle^{-\frac{k-1}{q}},$$

$(\sigma(\langle D_t\rangle) = (1 + |\tau|^2)^{1/2})$ then for $v \in C_0^\infty$ and of small support near $(0, 0)$ we have the estimate, following [11] and [6],

$$\sum\nolimits_{j=1}^{4} \|W_j v\|^2 \lesssim |\langle Pv, v\rangle| + \|v\|_{-\infty}^2$$

where, unless otherwise noted, norms and inner products are in $L^2(R^2)$. Here the last norm indicates a Sobolev norm of arbitrarily negative order. This estimate was established in [11] without the norm of $t^\ell B$ (and without the term $B^* t^{2\ell} B$ in the operator) and our estimate follows at once in our setting.

A first observation is that we may work microlocally near the τ axis, since away from that axis (conically) the operator is elliptic.

A second observation is that no localization in space is necessary, since away from the origin $(0,0)$, we have estimates on both $\|Bv\|^2$ and $\|B^*v\|^2$, and hence the usual subellipticity (since $q-1$ brackets of B and B^* generate the 'missing' vector field $\dfrac{\partial}{\partial t}$).

Our aim will be to show that for a solution u of $Pu = f \in C^\infty$ and arbitrary N,

$$\left(\frac{\partial}{\partial t}\right)^N u \in L^2_{\text{loc}}.$$

To do this, we pick a Sobolev space to which the solution belongs, i.e., in view of the ellipticity of P away from the τ axis, we pick s_0 such that $\langle D_t\rangle^{-s_0} u \in L^2_{\text{loc}}$ and from now on all indices on norms will be in the variable t only.

Actually we will change our point of view somewhat and assume that the left-hand side of the *a priori* estimate is finite locally for u with norms reduced by s_0 and show that this is true with the norms reduced by only $s_0 - \delta$ for some (fixed) $\delta > 0$.

Taking $s_0 = 0$ for simplicity, we will assume that $\sum_1^4 \|W_j u\|_0 < \infty$ and show that in fact $\sum_1^4 \|W_j u\|_\delta < \infty$ for some positive δ. Iterating this 'bootstrap operation will prove that the solution is indeed smooth.

The main new ingredient in proving hypoellipticity is the presence of the term $t^\ell B$, which will result in new brackets. As in Kohn's work and ours, the solution u will initially be smoothed out in t so that the estimate may be applied freely, and at the end the smoothing will be allowed to tend suitably to the identity and we will be able to apply a Lebesgue bounded convergence theorem to show that the $\sum_1^4 \|W_j u\|_\delta$ are also finite, leading to hypoellipticity.

Without loss of generality, as observed above, we may assume that the solution u to $Pu = f \in C_0^\infty$ has small support near the origin (to be more thorough, we could take a localizing function of small support, ζ, and write $P\zeta u = \zeta P u + [P,\zeta]u = \zeta f \mod C_0^\infty$ so that $P\zeta u \in C_0^\infty$ since we have already seen that u will be smooth in the support of derivatives of ζ by the hypoellipticity of P away from the origin.)

In order to smooth out the solution in the variable t, we introduce a standard cut-off function $\chi(\tau) \in C_0^\infty(|\tau| \le 2), \chi(\tau) \equiv 1, |\tau| \le 1$, and set $\chi_M(\tau) = \chi(\tau/M)$. Thus $\chi_M(D)$ is infinitely smoothing (in t) and, in supp $\chi'_M, \tau \sim M$ and $|\chi_M^{(j)}| \sim M^{-j}$. Further, as $M \to \infty$, $\chi_M(D) \to Id$ in such a way that it suffices to show $\|\chi_M(D)w\|_r \le C$ independent of M to conclude that $w \in H^r$.

Introducing of χ_M, however, destroys compact support, so we shall introduce $v = \psi(x,t)\langle D_t\rangle^\delta \chi_M(D)u$ into the *a priori* estimate and show that the left-hand remains bounded uniformly in M as $M \to \infty$.

For clarity, we restate the estimate in the form in which we will use it, suppressing the spatial localization now as discussed above:

$$\|B^*\langle D_t\rangle^\delta \chi_M u\|^2 + \|t^\ell B\langle D_t\rangle^\delta \chi_M u\|^2 + \|x^k B\langle D_t\rangle^\delta \chi_M u\|^2 + \|\langle D_t\rangle^{-\frac{k-1}{q}}\langle D_t\rangle^\delta \chi_M u\|^2$$
$$\lesssim (P\langle D_t\rangle^\delta \chi_M u, \langle D_t\rangle^\delta \chi_M u) + \|\langle D_t\rangle^\delta \chi_M u\|^2_{-\infty}.$$

Clearly the most interesting bracket which will enter in bringing $\langle D_t\rangle^\delta \chi_M$ past the operator P, and the only term which has not been handled in the two papers cited above, is when t^l is differentiated, as in

$$([B^* t^{2\ell} B, \langle D_t\rangle^\delta \chi_M]u, \langle D_t\rangle^\delta \chi_M u) \sim (B^*[t^{2\ell}, \langle D_t\rangle^\delta \chi_M]Bu, \langle D_t\rangle^\delta \chi_M u)$$
$$\sim \sum (t^{2\ell-j}(\langle D_t\rangle^\delta \chi_M)^{(j)}Bu, B\langle D_t\rangle^\delta \chi_M u)$$

in obvious notation. Here the derivatives on the symbol of $\langle D_t\rangle^\delta \chi_M$ are denoted $(\langle D_t\rangle^\delta \chi_M)^{(j)}$.

So a typical term would lead, after using a weighted Schwarz inequality and absorbing a term on the left-hand side of the estimate, to the need to estimate a constant times the norm

$$\|t^{\ell-j}(\langle D_t\rangle^\delta \chi_M)^{(j)}Bu\|^2.$$

Now we are familiar with handling such terms, although in the above-cited works it was powers of x (or z in the complex case) instead of powers of t. The method employed is to 'raise and lower' powers of t and of τ on one side of an inner product and lower them on the other. That is, if we denote by A the operator

$$A = t\langle D_t\rangle^\rho,$$

we have

$$\|A^r w\|^2 = |(A^\rho w, A^\rho w)| \lesssim_N \|w\|^2 + \|A^N w\|^2$$

for any desired positive $N \geq r$ (repeated integrations by parts or by interpolation, since the non-self-adjointness of A is of lower order), together with the observation that a small constant may be placed in front of either term on the right, and the notation $\lesssim_N$ means that the constants involved may depend on N, but N will always be bounded.

In our situation, looking first at the case $j = 1$,

$$\begin{aligned}
&\|t^{\ell-1}B(\langle D_t\rangle^\delta \chi_M)'u\|^2 \\
&\quad = |(A^{\ell-1}\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u, A^{\ell-1}\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u)| \\
&\quad \leq \|A^\ell\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u\|^2 + \|\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u\|^2 \\
&\quad = \|t^\ell\langle D_t\rangle^\rho B(\langle D_t\rangle^\delta \chi_M)'u\|^2 + \|\langle D_t\rangle^{-(\ell-1)\rho}B(\langle D_t\rangle^\delta \chi_M)'u\|^2 \\
&\quad \sim \|t^\ell B\tilde\chi_M u\|^2_{\rho+\delta-1} + \|B\tilde\chi_M u\|^2_{-(\ell-1)\rho+\delta-1}
\end{aligned}$$

modulo further brackets, where $\tilde{\chi}_M$ is another function of τ such as $\langle D_t\rangle\chi'_M$, with symbol uniformly bounded in τ independently of M and of compact support. $\tilde{\chi}_M$ will play the same role as χ_M in future iterations of the *a priori* estimate.

We are not yet done – the first term on the right will be handled inductively provided $\rho - 1 < 0$, but the second contains just B without the essential powers of t.

However, as in [11], we may integrate by parts, thereby converting B to B^* which is maximally controlled in the estimate, but modulo a term arising from the bracket of B and B^*.

As in [6] or [11], or by direct computation, we have

$$\|Bw\|_r^2 \lesssim \|B^*w\|_r^2 + \|x^{\frac{q-2}{2}}w\|_{r+1/2}$$

and while this power of x may not be directly useful, we confronted the same issue in [6] (in the complex form – the 'real' one is analogous). In that context, the exponent $q-2/2$ was denoted $m-1$, but the term was well estimated in norm $-\frac{1}{2m}+\frac{1}{2}-\frac{k-1}{q}$, which in this context reads $-\frac{1}{q}+\frac{1}{2}-\frac{k-1}{q}=\frac{1}{2}-\frac{k}{q}$. We have $-(\ell-1)\rho-1+\frac{1}{2}$, and under our hypothesis that $\ell > k/q$ our norm is less than $1/2-k/q$ for any choice of $\rho \leq 1$ as desired.

Finally, the terms with $j > 1$ work out similarly.

This means that we do indeed have a weaker norm so that with a different cut off in τ, which we have denoted $\tilde{\chi}_M$, there is a gain, and that as $M \to \infty$ this term will remain bounded.

3. Gevrey hypoellipticity

Again we write the example as

$$P = \sum_1^3 W_j^* W_j,$$

with

$$W_1 = B^*, \quad W_2 = t^\ell B, \quad W_3 = x^k B, \quad B = D_x + ix^{q-1}D_t$$

and omit localization as discussed above, and set $v = T^p u$, the *a priori* estimate we have is

$$\sum_1^4 \|W_j v\|_0^2 \lesssim |(Pv, v)|, \quad W_4 = \langle D_t\rangle^{-\frac{k-1}{q}}.$$

The principal (bracketing) errors come from $[W_j, T^p]v, j = 1, 2, 3,$ and the worst case occurs when $j = 2$:

$$[W_2, T^p]v = p\ell t^{\ell-1}BT^{p-1}v.$$

Raising and lowering powers of t as above,

$$\begin{aligned}\|t^{\ell-1}BT^{p-1}u\| &\lesssim \|t^{\ell}BT^{p-1+\delta}u\| + \|BT^{p-1-(\ell-1)\delta}u\| \\ &\lesssim \|W_2T^{p-1+\delta}u\| + \|B^*T^{p-1-(\ell-1)\delta}u\| + \|x^{q-1}T^{p-(\ell-1)\delta}u\|\end{aligned}$$

using the fact that $B - B^* = \pm ix^{q-1}T$. Again we raise and lower powers of x to obtain

$$\|x^{q-1}T^{p-(\ell-1)\delta}u\| \lesssim \|T^{p-(\ell-1)\delta-(q-1)\rho}u\| + \|\{x^{k+q-1}T\}\,T^{p-1-(\ell-1)\delta+k\rho}\,u\|$$

since the term in braces is a linear combination of x^kB and B^*, both of which are optimally estimated. The result is that

$$\begin{aligned}\|t^{\ell-1}BT^{p-1}u\| &\lesssim \|t^{\ell}BT^{p-1+\delta}u\| + \|BT^{p-1-(\ell-1)\delta}u\| \\ &\lesssim \|W_2T^{p-1+\delta}u\| + \|W_1T^{p-1-(\ell-1)\delta}u\| + \|W_1T^{p-1-(\ell-1)\delta+k\rho}\,u\| \\ &\quad + \|W_4T^{p-(\ell-1)\delta-(q-1)\rho+\frac{k-1}{\ell}}u\| + \|W_3T^{p-1-(\ell-1)\delta+k\rho}\,u\|\end{aligned}$$

where the third term on the right clearly dominates the second. In all, then,

$$\|t^{\ell-1}BT^{p-1}u\| \lesssim \sum_{j=1}^{4} \|X_jT^{p-\sigma}u\|$$

where

$$\sigma = \min_i \sup_{\substack{0<\rho<1 \\ 0<\delta<1}} s_i$$

with

$$\begin{aligned}s_1 &= 1-\delta \\ s_2 &= (\ell-1)\delta + (q-1)\rho - \frac{l-1}{q} \\ s_3 &= 1 + (\ell-1)\delta - k\rho.\end{aligned}$$

The desired value of σ is achieved when all three are equal by a standard minimax argument, and this occurs when

$$\delta = \frac{k}{q\ell}, \qquad \rho = \frac{1}{q}$$

resulting in $\sigma = \dfrac{\ell q - k}{\ell q}$, which yields $G^{\frac{\ell q}{\ell q-k}} = G^{1+\frac{k}{\ell q-k}}$ hypoellipticity.

The restriction that $k < \ell q$ for hypoellipticity at all takes on greater meaning given this result.

4. Computing Λ

4.1. q-pseudodifferential calculus

The idea, attributed by J. Sjöstrand and Zworski [13] to Schur, is essentially a linear algebra remark: assume that the $n \times n$ matrix A has zero in its spectrum with multiplicity one. Then of course A is not invertible, but, denoting by e_0 the zero eigenvector of A, the matrix (in block form)

$$\begin{bmatrix} A & e_0 \\ {}^t e_0 & 0 \end{bmatrix}$$

is invertible as a $(n+1) \times (n+1)$ matrix in $\mathbb{C}^{n+1}$. Here ${}^t e_0$ denotes the row vector e_0.

All we want to do is to apply this remark to the operator P whose part BB^* has the same problem as the matrix A, i.e., a zero simple eigenvalue. This occurs since q is even. Note that in the case of odd q, P may easily be seen to be hypoelliptic.

It is convenient to use self-adjoint derivatives from now on, so that the vector field $B^* = D_x - ix^{q-1}D_t$, where $D_x = i^{-1}\partial_x$. It will also be convenient to write $B(x, \xi, \tau)$ for the symbol of the vector field B, i.e., $B(x, \xi, \tau) = \xi + ix^{q-1}\tau$ and analogously for the other vector fields involved. The symbol of P can be written as

$$P(x, t, \xi, \tau) = P_0(x, t, \xi, \tau) + P_{-q}(t, x, \xi, \tau) + P_{-2k}(x, t, \xi, \tau), \tag{4.1.1}$$

where

$$P_0(x, t, \xi, \tau) = (1 + t^{2\ell})(\xi^2 + x^{2(q-1)}\tau^2) + (-1 + t^{2\ell})(q-1)x^{q-2}\tau;$$

$$P_{-q}(x, t, \xi, \tau) = -2\ell t^{2\ell-1}x^{q-1}(\xi + ix^{q-1}\tau);$$

$$P_{-2k}(x, t, \xi, \tau) = x^{2k}(\xi^2 + x^{2(q-1)}\tau^2) - i2kx^{2k-1}(\xi + ix^{q-1}\tau) + (q-1)x^{2k+q-2}\tau.$$

It is evident at a glance that the different pieces into which P has been decomposed include terms of different order and vanishing speed. We thus need to say something about the adopted criteria for the above decomposition.

Let μ be a positive number and consider the following canonical dilation in the variables (x, t, ξ, τ):

$$x \to \mu^{-1/q}x, \qquad t \to t, \qquad \xi \to \mu^{1/q}\xi, \qquad \tau \to \mu\tau.$$

It is then evident that P_0 has then the following homogeneity property

$$P_0(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau) = \mu^{2/q}P_0(x, t, \xi, \tau). \tag{4.1.2}$$

Analogously

$$P_{-q}(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau) = \mu^{2/q-1}P_q(x, t, \xi, \tau) \tag{4.1.3}$$

and

$$P_{-2k}(\mu^{-1/q}x, t, \mu^{1/q}\xi, \mu\tau) = \mu^{2/q-(2k)/q}P_{2k}(x, t, \xi, \tau). \tag{4.1.4}$$

Now the above homogeneity properties help us in identifying some symbol classes suitable for P.

Following the ideas of [1] and [2] we define the following class of symbols:

Definition 4.1.1. We define the class of symbols $S_q^{m,k}(\Omega, \Sigma)$ where Ω is a conic neighborhood of the point $(0, e_2)$ and Σ denotes the characteristic manifold $\{x = 0, \xi = 0\}$, as the set of all C^∞ functions such that, on any conic subset of Ω with compact base,

$$\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma \partial_\xi^\delta a(x,t,\xi,\tau)\right| \lesssim (1+|\tau|)^{m-\beta-\delta} \left(\frac{|\xi|}{|\tau|} + |x|^{q-1} + \frac{1}{|\tau|^{\frac{q-1}{q}}}\right)^{k-\frac{\gamma}{q-1}-\delta}. \tag{4.1.5}$$

We write $S_q^{m,k}$ for $S_q^{m,k}(\mathbb{R}^2 \times \mathbb{R}^2, \Sigma)$.

By a straightforward computation, see, e.g., [2], we have $S_q^{m,k} \subset S_q^{m',k'}$ iff $m \leq m'$ and $m - \frac{q-1}{q}k \leq m' - \frac{q-1}{q}k'$. $S_q^{m,k}$ can be embedded in the Hörmander classes $S_{\rho,\delta}^{m+\frac{q-1}{q}k_-}$, where $k_- = \max\{0, -k\}$, $\rho = \delta = 1/q \leq 1/2$. Thus we immediately deduce that $P_0 \in S_q^{2,2}$, $P_{-q} \in S_q^{1,2} \subset S_q^{2,2+\frac{q}{q-1}}$ and finally $P_{-2k} \in S_q^{2,2+\frac{2k}{q-1}}$.

We shall need also the following

Definition 4.1.2 ([2]). Let Ω and Σ be as above. We define the class $\mathscr{H}_q^m(\Omega, \Sigma)$ by

$$\mathscr{H}_q^m(\Omega, \Sigma) = \cap_{j=1}^\infty S_q^{m-j, -\frac{q}{q-1}j}(\Omega, \Sigma).$$

We write $\mathscr{H}_q^m$ for $\mathscr{H}_q^m(\mathbb{R}^2 \times \mathbb{R}^2, \Sigma)$.

Now it is easy to see that P_0, as a differential operator w.r.t. the variable x, depending on the parameters t, $\tau \geq 1$ has a non negative discrete spectrum. Moreover the dependence on τ of the eigenvalue is particularly simple, because of (4.1.2). Call $\Lambda_0(t,\tau)$ the lowest eigenvalue of P_0. Then

$$\Lambda_0(t,\tau) = \tau^{\frac{2}{q}} \tilde{\Lambda}_0(t).$$

Moreover Λ_0 has multiplicity one and $\tilde{\Lambda}_0(0) = 0$, since BB^* has a null eigenvalue with multiplicity one. Denote by $\varphi_0(x,t,\tau)$ the corresponding eigenfunction. Because of (4.1.2), we have the following properties of φ_0:

a) For fixed $(t,\tau), \varphi_0$ is exponentially decreasing w.r.t. x as $x \to \pm\infty$. In fact, because of (4.1.2), setting $y = x\tau^{1/q}$, we have that $\varphi_0(y,t,\tau) \sim e^{-y^q/q}$.
b) It is convenient to normalize φ_0 in such a way that $\|\varphi_0(\cdot,t,\tau)\|_{L^2(\mathbb{R}_x)} = 1$. This implies that a factor $\sim \tau^{1/2q}$ appears. Thus we are led to the definition of a Hermite operator (see [9] for more details).

Let $\Sigma_1 = \pi_x \Sigma$ be the space projection of Σ. Then we write

Definition 4.1.3. We write H_q^m for $\mathscr{H}_q^m(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$, i.e., the class of all smooth functions in $\cap_{j=1}^\infty S_q^{m-j,-\frac{q}{q-1}j}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$. Here $S_q^{m,k}(\mathbb{R}^2_{x,t} \times \mathbb{R}_\tau, \Sigma_1)$ denotes the set of all smooth functions such that

$$\left|\partial_t^\alpha \partial_\tau^\beta \partial_x^\gamma a(x,t,\tau)\right| \lesssim (1+|\tau|)^{m-\beta} \left(|x|^{q-1} + \frac{1}{|\tau|^{\frac{q-1}{q}}}\right)^{k-\frac{\gamma}{q-1}}. \tag{4.1.6}$$

Define the action of a symbol $a(x,t,\tau)$ in H_q^m as the map $a(x,t,D_t)\colon C_0^\infty(\mathbb{R}_t) \longrightarrow C^\infty(\mathbb{R}^2_{x,t})$ defined by

$$a(x,t,D_t)u(x,t) = (2\pi)^{-1}\int e^{it\tau}a(x,t,\tau)\hat{u}(\tau)d\tau.$$

Such an operator, modulo a regularizing operator (w.r.t. the t variable) is called a Hermite operator and we denote by OPH_q^m the corresponding class.

We need also the adjoint of the Hermite operators defined in Definition 4.1.3.

Definition 4.1.4. Let $a \in H_q^m$. We define the map $a^*(x,t,D_t)\colon C_0^\infty(\mathbb{R}^2_{x,t}) \longrightarrow C^\infty(\mathbb{R}_t)$ as

$$a^*(x,t,D_t)u(t) = (2\pi)^{-1}\int\int e^{it\tau}\overline{a(x,t,\tau)}\hat{u}(x,\tau)dxd\tau.$$

We denote by OPH_q^{*m} the related set of operators.

Lemma 4.1.1. *Let $a \in H_q^m$, $b \in S_q^{m,k}$; then*

(i) *the formal adjoint $a(x,t,D_t)^*$ belongs to OPH_q^{*m} and its symbol has the asymptotic expansion*

$$\sigma(a(x,t,D_t)^*) - \sum_{\alpha=0}^{N-1}\frac{1}{\alpha!}\partial_\tau^\alpha D_t^\alpha\overline{a(x,t,\tau)} \in H_q^{m-N}. \tag{4.1.7}$$

(ii) *The formal adjoint $(a^*(x,t,D_t))^*$ belongs to OPH_q^m and its symbol has the asymptotic expansion*

$$\sigma(a^*(x,t,D_t)^*) - \sum_{\alpha=0}^{N-1}\frac{1}{\alpha!}\partial_\tau^\alpha D_t^\alpha a(x,t,\tau) \in H_q^{m-N}. \tag{4.1.8}$$

(iii) *The formal adjoint $b(x,t,D_x,D_t)^*$ belongs to $OPS_q^{m,k}$ and its symbol has the asymptotic expansion*

$$\sigma(a(x,t,D_x,D_t)^*) - \sum_{\alpha=0}^{N-1}\frac{1}{\alpha!}\partial_{(\xi,\tau)}^\alpha D_{(x,t)}^\alpha\overline{a(x,t,\xi,\tau)} \in S_q^{m-N,k-N\frac{q}{q-1}}. \tag{4.1.9}$$

The following is a lemma on compositions involving the two different types of Hermite operators defined above. First we give a definition of “global” homogeneity:

Definition 4.1.5. We say that a symbol $a(x,t,\xi,\tau)$ is globally homogeneous (abbreviated g.h.) of degree m, if, for $\lambda \geq 1$, $a(\lambda^{-1/q}x,t,\lambda^{1/q}\xi,\lambda\tau) = \lambda^m a(x,t,\xi,\tau)$. Analogously a symbol, independent of ξ, of the form $a(x,t,\tau)$ is said to be globally homogeneous of degree m if $a(\lambda^{-1/q}x,t,\lambda\tau) = \lambda^m a(x,t,\tau)$.

Let $f_{-j}(x,t,\xi,\tau) \in S_q^{m,k+\frac{j}{q-1}}$, $j \in \mathbb{N}$, then there exists $f(x,t,\xi,\tau) \in S_q^{m,k}$ such that $f \sim \sum_{j\geq 0} f_{-j}$, i.e., $f - \sum_{j=0}^{N-1} f_{-j} \in S_q^{m,k+\frac{N}{q-1}}$, thus f is defined modulo a symbol in $S_q^{m,\infty} = \cap_{h\geq 0}S_q^{m,h}$.

Analogously, let f_{-j} be globally homogeneous of degree $m - k\frac{q-1}{q} - \frac{j}{q}$ and such that for every $\alpha, \beta \geq 0$ satisfies the estimates

$$\left|\partial^{\gamma}_{(t,\tau)}\partial^{\alpha}_{x}\partial^{\beta}_{\xi} f_{-j}(x,t,\xi,\tau)\right| \lesssim \left(|\xi| + |x|^{q-1} + 1\right)^{k-\frac{\alpha}{q-1}-\beta}, \qquad (x,\xi) \in \mathbb{R}^2, \quad (4.1.10)$$

for (t,τ) in a compact subset of $\mathbb{R} \times \mathbb{R} \setminus 0$ and every multiindex γ. Then $f_{-j} \in S_q^{m,k+\frac{j}{q-1}}$.

Accordingly, let $\varphi_{-j}(x,t,\tau) \in H_q^{m-\frac{j}{q}}$, then there exists $\varphi(x,t,\tau) \in H_q^m$ such that $\varphi \sim \sum_{j\geq 0} \varphi_{-j}$, i.e., $\varphi - \sum_{j=0}^{N-1} \varphi_{-j} \in H_q^{m-\frac{N}{q}}$, so that φ is defined modulo a symbol regularizing (w.r.t. the t variable.)

Similarly, let φ_{-j} be globally homogeneous of degree $m - \frac{j}{q}$ and such that for every $\alpha, \ell \geq 0$ satisfies the estimates

$$\left|\partial^{\beta}_{(t,\tau)}\partial^{\alpha}_{x}\varphi_{-j}(x,t,\tau)\right| \lesssim \left(|x|^{q-1} + 1\right)^{-\ell-\frac{\alpha}{q-1}}, \qquad x \in \mathbb{R}, \qquad (4.1.11)$$

for (t,τ) in a compact subset of $\mathbb{R} \times \mathbb{R} \setminus 0$ and every multiindex β. Then $\varphi_{-j} \in H_q^{m-\frac{j}{q}}$.

As a matter of fact in the construction below we deal with asymptotic series of homogeneous symbols.

Next we give a brief description of the composition of the various types of operator introduced so far.

Lemma 4.1.2 ([9], Formula 2.4.9). *Let $a \in S_q^{m,k}$, $b \in S_q^{m',k'}$, with asymptotic globally homogeneous expansions*

$$\begin{aligned} a &\sim \sum_{j\geq 0} a_{-j}, \qquad a_{-j} \in S_q^{m,k+\frac{j}{q-1}}, \text{ g.h. of degree } m - \frac{q-1}{q}k - \frac{j}{q} \\ b &\sim \sum_{i\geq 0} b_{-i}, \qquad b_{-i} \in S_q^{m',k'+\frac{i}{q-1}}, \text{ g.h. of degree } m' - \frac{q-1}{q}k' - \frac{i}{q}. \end{aligned}$$

Then $a \circ b$ is an operator in $OPS_q^{m+m',k+k'}$ with

$$\sigma(a\circ b) - \sum_{s=0}^{N-1} \sum_{q\alpha+i+j=s} \frac{1}{\alpha!}\sigma\left(\partial^{\alpha}_{\tau} a_{-j}(x,t,D_x,\tau) \circ_x D^{\alpha}_{t} b_{-i}(x,t,D_x,\tau)\right) \in S_q^{m+m'-N,k+k'}. \quad (4.1.12)$$

Here $\circ_x$ denotes the composition w.r.t. the x-variable.

Lemma 4.1.3 ([2], Section 5 and [9], Sections 2.2, 2.3). *Let* $a \in H_q^m$, $b \in H_q^{m'}$ *and* $\lambda \in S_{1,0}^{m''}(\mathbb{R}_t \times \mathbb{R}_\tau)$ *with homogeneous asymptotic expansions*

$$\begin{aligned} a &\sim \sum_{j\geq 0} a_{-j}, \qquad a_{-j} \in H_q^{m-\frac{j}{q}}, \text{ g.h. of degree } m - \frac{j}{q} \\ b &\sim \sum_{i\geq 0} b_{-i}, \qquad b_{-i} \in H_q^{m'-\frac{i}{q}}, \text{ g.h. of degree } m' - \frac{i}{q} \\ \lambda &\sim \sum_{\ell\geq 0} \lambda_{-\ell}, \qquad \lambda_{-\ell} \in S_{1,0}^{m''-\frac{\ell}{q}}, \text{ homogeneous of degree } m'' - \frac{\ell}{q} \end{aligned}$$

Then

(i) $a \circ b^*$ *is an operator in* $OP\mathscr{H}_q^{m+m'-\frac{1}{q}}(\mathbb{R}^2, \Sigma)$ *with*

$$\sigma(a \circ b^*)(x,t,\xi,\tau) - e^{-ix\xi} \sum_{s=0}^{N-1} \sum_{q\alpha+i+j=s} \frac{1}{\alpha!} \partial_\tau^\alpha a_{-j}(x,t,\tau) D_t^\alpha \hat{\bar{b}}_{-i}(\xi,t,\tau) \in \mathscr{H}_q^{m+m'-\frac{1}{q}-\frac{N}{q}}, \quad (4.1.13)$$

where the Fourier transform in $D_t^\alpha \hat{\bar{b}}_{-i}(\xi,t,\tau)$ *is taken w.r.t. the* x*-variable.*

(ii) $b^* \circ a$ *is an operator in* $OPS_{1,0}^{m+m'-\frac{1}{q}}(\mathbb{R}_t)$ *with*

$$\sigma(b^* \circ a)(t,\tau) - \sum_{s=0}^{N-1} \sum_{q\alpha+j+i=s} \frac{1}{\alpha!} \int \partial_\tau^\alpha \bar{b}_{-i}(x,t,\tau) D_t^\alpha a_{-j}(x,t,\tau) dx \in S_{1,0}^{m+m'-\frac{1}{q}-\frac{N}{q}}(\mathbb{R}_t). \quad (4.1.14)$$

(iii) $a \circ \lambda$ *is an operator in* $OPH_q^{m+m''}$. *Furthermore its asymptotic expansion is given by*

$$\sigma(a \circ \lambda) - \sum_{s=0}^{N-1} \sum_{q\alpha+j+\ell=s} \frac{1}{\alpha!} \partial_\tau^\alpha a_{-j}(x,t,\tau) D_t^\alpha \lambda_{-\ell}(t,\tau) \in H_q^{m+m''-\frac{N}{q}}. \qquad (4.1.15)$$

Lemma 4.1.4. *Let* $a(x,t,D_x,D_t)$ *be an operator in the class* $OPS_q^{m,k}(\mathbb{R}^2,\Sigma)$ *and* $b(x,t,D_t) \in OPH_q^{m'}$ *with g.h. asymptotic expansions*

$$\begin{aligned} a &\sim \sum_{j\geq 0} a_{-j}, \qquad a_{-j} \in S_q^{m,k+\frac{j}{q-1}}, \text{ g.h. of degree } m - \frac{q-1}{q}k - \frac{j}{q} \\ b &\sim \sum_{i\geq 0} b_{-i}, \qquad b_{-i} \in H_q^{m'-\frac{i}{q-1}}, \text{ g.h. of degree } m' - \frac{i}{q}. \end{aligned}$$

Then $a \circ b \in OPH_q^{m+m'-k\frac{q-1}{q}}$ *and has a g.h. asymptotic expansion of the form*

$$\sigma(a \circ b) - \sum_{s=0}^{N-1} \sum_{q\ell+i+j=s} \frac{1}{\ell!} \partial_\tau^\ell a_{-j}(x,t,D_x,\tau)(D_t^\ell b_{-i}(\cdot,t,\tau)) \in H_q^{m+m'-k\frac{q-1}{q}-\frac{N}{q}}. \quad (4.1.16)$$

Lemma 4.1.5. *Let* $a(x,t,D_x,D_t)$ *be an operator in the class* $OPS_q^{m,k}(\mathbb{R}^2,\Sigma)$, $b^*(x,t,D_t) \in OPH_q^{*m'}$ *and* $\lambda(t,D_t) \in OPS_{1,0}^{m''}(\mathbb{R}_t)$ *with homogeneous asymptotic expansions*

$$\begin{aligned} a &\sim \sum_{j\geq 0} a_{-j}, \qquad a_{-j} \in S_q^{m,k+\frac{j}{q-1}}, \text{ g.h. of degree } m - \frac{q-1}{q}k - \frac{j}{q} \\ b &\sim \sum_{i\geq 0} b_{-i}, \qquad b_{-i} \in H_q^{m'-\frac{i}{q-1}}, \text{ g.h. of degree } m' - \frac{i}{q} \\ \lambda &\sim \sum_{\ell\geq 0} \lambda_{-\ell}, \qquad \lambda_{-\ell} \in S_{1,0}^{m''-\frac{\ell}{q}}, \text{ homogeneous of degree } m'' - \frac{\ell}{q} \end{aligned}$$

Then

(i) $b^*(x,t,D_t) \circ a(x,t,D_x,D_t) \in OPH_q^{*m+m'-\frac{q-1}{q}k}$ *with g.h. asymptotic expansion*

$$\sigma(b^* \circ a) - \sum_{s=0}^{N-1} \sum_{q\ell+i+j=s} \frac{1}{\ell!} D_t^\ell \left(\overline{a_{-j}}(x,t,D_x,\tau)\right)^* \left(\partial_\tau^\ell \overline{b_{-i}}(\cdot,t,\tau)\right) \in H_q^{m+m'-k\frac{q-1}{q}-\frac{N}{q}}. \quad (4.1.17)$$

(ii) $\lambda(t,D_t) \circ b^*(x,t,D_t) \in OPH_q^{*m'+m''}$ *with asymptotic expansion*

$$\sigma(\lambda \circ b^*) - \sum_{s=0}^{N-1} \sum_{q\alpha+i+\ell=s} \frac{1}{\alpha!} \partial_\tau^\alpha \lambda_{-\ell}(t,\tau) D_t^\alpha \overline{b_{-i}}(x,t,\tau) \in H_q^{m'+m''-\frac{N}{q}}. \quad (4.1.18)$$

The proofs of Lemmas 4.1.2–4.1.4 are obtained with a q-variation of the calculus developed by Boutet de Monvel and Helffer, [2], [9]. The proof of Lemma 4.1.5 is performed taking the adjoint and involves a combinatoric argument; we give here a sketchy proof.

Proof. We prove item (i). The proof of (ii) is similar and simpler.

Since $b^*(x,t,D_t) \circ a(x,t,D_x,D_t) = (a(x,t,D_x,D_t)^* \circ b^*(x,t,D_t)^*)^*$, using Lemma 4.1.1 and 4.1.2, we first compute

$$\begin{aligned} &\sigma(a(x,t,D_x,D_t)^* \circ b^*(x,t,D_t)^*) \\ &\quad = \sum_{\alpha,\ell,p,i,j\geq 0} \frac{1}{\ell!\alpha!p!} \partial_\tau^{\alpha+p} D_t^\alpha \left(a_{-j}(x,t,D_x,\tau)\right)^* \left(\partial_\tau^\ell D_t^{\ell+p} b_{-i}(\cdot,t,\tau)\right) \end{aligned}$$

$$= \sum_{\gamma\geq 0} \frac{1}{\gamma!} \partial_\tau^\gamma D_t^\gamma \left(\sum_{\beta,i,j\geq 0} \frac{1}{\beta!} (-D_t)^\beta \left(a_{-j}(x,t,D_x,\tau)\right)^* \left(\partial_\tau^\beta b_{-i}(\cdot,t,\tau)\right) \right),$$

where $(-D_t)^\beta \left(a_{-j}(x,t,D_x,\tau)\right)^*$ denotes the formal adjoint of the operator with symbol $D_t^\beta a_{-j}(x,t,\xi,\tau)$ as an operator in the x-variable, depending on (t,τ) as parameters. Here we used Formula (A.2) in the Appendix. Hence

$$\sigma(b^*(x,t,D_t) \circ a(x,t,D_x,D_t)) = \sum_{\ell\geq 0} \frac{1}{\ell!} \partial_\tau^\ell D_t^\ell$$

$$\times \left(\sum_{\gamma\geq 0} \frac{1}{\gamma!} \partial_\tau^\gamma D_t^\gamma \left(\sum_{\beta,i,j\geq 0} \frac{1}{\beta!} (-D_t)^\beta \left(a_{-j}(x,t,D_x,\tau)\right)^* \left(\partial_\tau^\beta b_{-i}(\cdot,t,\tau)\right) \right) \right)^-$$

$$= \sum_{\beta,i,j\geq 0} \frac{1}{\beta!} D_t^\beta \left(\overline{a_{-j}}(x,t,D_x,\tau)\right)^* \left(\partial_\tau^\beta \overline{b_{-i}}(\cdot,t,\tau)\right)$$

$$= \sum_{s\geq 0} \sum_{q\beta+i+j=s} \frac{1}{\beta!} D_t^\beta \left(\overline{a_{-j}}(x,t,D_x,\tau)\right)^* \left(\partial_\tau^\beta \overline{b_{-i}}(\cdot,t,\tau)\right),$$

because of Formula (A.3) of the Appendix. □

4.2. The actual computation of the eigenvalue

We are now in a position to start computing the symbol of Λ.

Let us first examine the minimum eigenvalue and the corresponding eigenfunction of $P_0(x,t,D_x,\tau)$ in (4.1.1), as an operator in the x-variable. It is well known that $P_0(x,t,D_x,\tau)$ has a discrete set of non negative, simple eigenvalues depending in a real analytic way on the parameters (t,τ).

P_0 can be written in the form $LL^* + t^{2\ell} L^* L$, where $L = D_x + ix^{q-1}\tau$. The kernel of L^* is a one-dimensional vector space generated by $\varphi_{0,0}(x,\tau) = c_0 \tau^{\frac{1}{2q}} \exp(-\frac{x^q}{q}\tau)$, c_0 being a normalization constant such that $\|\varphi_{0,0}(\cdot,\tau)\|_{L^2(\mathbb{R}_x)} = 1$. We remark that in this case τ is positive. For negative values of τ the operator LL^* is injective. Denoting by $\varphi_0(x,t,\tau)$ the eigenfunction of P_0 corresponding to its lowest eigenvalue $\Lambda_0(t,\tau)$, we obtain that $\varphi_0(x,0,\tau) = \varphi_{0,0}(x,\tau)$ and that $\Lambda_0(0,\tau) = 0$. As a consequence the operator

$$P = BB^* + B^*(t^{2\ell} + x^{2k})B, \qquad B = D_x + ix^{q-1} D_t, \tag{4.2.1}$$

is not maximally hypoelliptic, i.e., hypoelliptic with a loss of $2 - \frac{2}{q}$ derivatives.

Next we give a more precise description of the t-dependence of both the eigenvalue Λ_0 and its corresponding eigenfunction φ_0 of $P_0(x,t,D_x,\tau)$.

It is well known that there exists an $\varepsilon > 0$, small enough, such that the operator

$$\Pi_0 = \frac{1}{2\pi i} \oint_{|\mu|=\varepsilon} (\mu I - P_0(x,t,D_x,\tau))^{-1} d\mu$$

is the orthogonal projection onto the eigenspace generated by φ_0. Note that Π_0 depends on the parameters (t, τ). The operator LL^* is thought of as an unbounded operator in $L^2(\mathbb{R}_x)$ with domain $B_q^2(\mathbb{R}_x) = \{u \in L^2(\mathbb{R}_x) \mid x^\alpha D_x^\beta u \in L^2, 0 \le \beta + \frac{\alpha}{q-1} \le 2\}$. We have

$$(\mu I - P_0)^{-1} = \left(I + t^{2\ell}\left[-A(I + t^{2\ell}A)^{-1}\right]\right)(\mu I - LL^*)^{-1},$$

where $A = (LL^* - \mu I)^{-1} L^* L$. Plugging this into the formula defining Π_0, we get

$$\Pi_0 = \frac{1}{2\pi i}\oint_{|\mu|=\varepsilon} (\mu I - LL^*)^{-1} d\mu - \frac{1}{2\pi i} t^{2\ell} \oint_{|\mu|=\varepsilon} A(I + t^{2\ell}A)^{-1}(\mu I - LL^*)^{-1} d\mu.$$

Hence

$$\begin{aligned}\varphi_0 = \Pi_0 \varphi_{0,0} &= \varphi_{0,0} - t^{2\ell}\frac{1}{2\pi i}\oint_{|\mu|=\varepsilon} A(I + t^{2\ell}A)^{-1}(\mu I - LL^*)^{-1}\varphi_{0,0} d\mu \\ &= \varphi_{0,0}(x, \tau) + t^{2\ell}\tilde{\varphi}_0(x, t, \tau). \end{aligned} \tag{4.2.2}$$

Since Π_0 is an orthogonal projection then $\|\varphi_0(\cdot, t, \tau)\|_{L^2(\mathbb{R}_x)} = 1$.

As a consequence we obtain that

$$\Lambda_0(t, \tau) = \langle P_0 \varphi_0, \varphi_0 \rangle = t^{2\ell}\|L\varphi_{0,0}\|^2 + \mathscr{O}(t^{4\ell}). \tag{4.2.3}$$

We point out that $L\varphi_{0,0} \neq 0$. Observe that, in view of (4.1.2),

$$\begin{aligned}\Lambda_0(t, \mu\tau) &= \min_{\substack{u \in B_q^2 \\ \|u\|_{L^2}=1}} \langle P_0(x, t, D_x, \mu\tau)u(x), u(x)\rangle \\ &= \min_{\substack{u \in B_q^2 \\ \|u\|_{L^2}=1}} \langle P_0(\mu^{-1/q}x, t, \mu^{1/q}D_x, \mu\tau)\frac{u(\mu^{-1/q}x)}{\mu^{-1/(2q)}}, \frac{u(\mu^{-1/q}x)}{\mu^{-1/(2q)}}\rangle \\ &= \mu^{\frac{2}{q}} \min_{\substack{v \in B_q^2 \\ \|v\|_{L^2}=1}} \langle P_0(x, t, D_x, \tau)v(x), v(x)\rangle \\ &= \mu^{\frac{2}{q}}\Lambda_0(t, \tau). \end{aligned} \tag{4.2.4}$$

This shows that Λ_0 is homogeneous of degree $2/q$ w.r.t. the variable τ.

Since φ_0 is the unique normalized solution of the equation $(P_0(x, t, D_x, \tau) - \Lambda_0(t, \tau))u(\cdot, t, \tau) = 0$, from (4.1.2) and (4.2.4) it follows that φ_0 is globally homogeneous of degree $1/(2q)$. Moreover φ_0 is rapidly decreasing w.r.t. the x-variable smoothly dependent on (t, τ) in a compact subset of $\mathbb{R}^2 \setminus 0$. Using estimates of the form (4.1.11) we can conclude that $\varphi_0 \in H_q^{1/(2q)}$.

Let us start now the construction of a right parametrix of the operator

$$\begin{bmatrix} P(x, t, D_x, D_t) & \varphi_0(x, t, D_t) \\ \varphi_0^*(x, t, D_t) & 0 \end{bmatrix}$$

as a map from $C_0^\infty(\mathbb{R}^2_{(x,t)}) \times C_0^\infty(\mathbb{R}_t)$ into $C^\infty(\mathbb{R}^2_{(x,t)}) \times C^\infty(\mathbb{R}_t)$. In particular we are looking for an operator such that

$$\begin{bmatrix} P(x,t,D_x,D_t) & \varphi_0(x,t,D_t) \\ \varphi_0^*(x,t,D_t) & 0 \end{bmatrix} \circ \begin{bmatrix} F(x,t,D_x,D_t) & \psi(x,t,D_t) \\ \psi^*(x,t,D_t) & -\Lambda(t,D_t) \end{bmatrix} \equiv \begin{bmatrix} Id_{C_0^\infty(\mathbb{R}^2)} & 0 \\ 0 & Id_{C_0^\infty(\mathbb{R})} \end{bmatrix}. \quad (4.2.5)$$

Here ψ and ψ^* denote operators in $OPH_q^{1/2q}$ and $OPH_q^{*\,1/2q}$, $F \in OPS_q^{-2,-2}$ and $\Lambda \in OPS_{1,0}^{2/q}$. Here $\equiv$ means equality modulo a regularizing operator.

From (4.2.5) we obtain four relations:

$$P(x,t,D_x,D_t) \circ F(x,t,D_x,D_t) + \varphi_0(x,t,D_t) \circ \psi^*(x,t,D_t) \equiv Id, \quad (4.2.6)$$

$$P(x,t,D_x,D_t) \circ \psi(x,t,D_t) - \varphi_0(x,t,D_t) \circ \Lambda(t,D_t) \equiv 0, \quad (4.2.7)$$

$$\varphi_0^*(x,t,D_t) \circ F(x,t,D_x,D_t) \equiv 0, \quad (4.2.8)$$

$$\varphi_0^*(x,t,D_t) \circ \psi(x,t,D_t) \equiv Id. \quad (4.2.9)$$

We are going to find the symbols F, ψ and Λ as asymptotic series of globally homogeneous symbols:

$$F \sim \sum_{j \geq 0} F_{-j}, \qquad \psi \sim \sum_{j\geq 0} \psi_{-j}, \qquad \Lambda \sim \sum_{j \geq 0} \Lambda_{-j}, \quad (4.2.10)$$

From Lemma 4.1.2 we obtain that

$$\sigma(P \circ F) \sim \sum_{s \geq 0} \sum_{q\alpha+i+j=s} \frac{1}{\alpha!} \sigma(\partial_\tau^\alpha P_{-j}(x,t,D_x,\tau) \circ_x D_t^\alpha F_{-i}(x,t,D_x,\tau)),$$

where we denoted by P_{-j} the globally homogeneous parts of degree $\frac{2}{q} - \frac{j}{q}$ of the symbol of P, so that $P = P_0 + P_{-q} + P_{-2k}$. Furthermore from Lemma 4.1.3(i) we may write that

$$\sigma(\varphi_0 \circ \psi^*) \sim e^{-ix\xi} \sum_{s\geq 0} \sum_{q\alpha+i=s} \frac{1}{\alpha!} \partial_\tau^\alpha \varphi_0(x,t,\tau) D_t^\alpha \hat{\bar{\psi}}_{-i}(\xi,t,\tau).$$

Analogously Lemmas 4.1.4, (4.1.3)(iii) give

$$\sigma(P \circ \psi) \sim \sum_{s\geq 0} \sum_{q\ell+i+j=s} \frac{1}{\ell!} \partial_\tau^\ell P_{-j}(x,t,D_x,\tau)(D_t^\ell \psi_{-i}(\cdot,t,\tau)),$$

$$\sigma(\varphi_0 \circ \Lambda) \sim \sum_{s \geq 0} \sum_{q\alpha+\ell=s} \frac{1}{\alpha!} \partial_\tau^\alpha \varphi_0(x,t,\tau) D_t^\alpha \Lambda_{-\ell}(t,\tau).$$

Finally Lemmas 4.1.5(i) and 4.1.3(ii) yield

$$\sigma(\varphi_0^* \circ F) \sim \sum_{s\geq 0} \sum_{q\ell+j=s} \frac{1}{\ell!} D_t^\ell \left(\overline{F_{-j}}(x,t,D_x,\tau)\right)^* (\partial_\tau^\ell \overline{\varphi_0}(\cdot,t,\tau)),$$

and

$$\sigma(\varphi_0^* \circ \psi) \sim \sum_{s\geq 0} \sum_{q\alpha+j=s} \frac{1}{\alpha!} \int \partial_\tau^\alpha \bar{\varphi}_0(x,t,\tau) D_t^\alpha \psi_{-j}(x,t,\tau) dx.$$

Let us consider the terms globally homogeneous of degree 0. We obtain the relations

$$P_0(x,t,D_x,\tau) \circ_x F_0(x,t,D_x,\tau) + \varphi_0(x,t,\tau) \otimes \psi_0(\cdot,t,\tau) = Id \quad (4.2.11)$$

$$P_0(x,t,D_x,\tau)(\psi_0(\cdot,t,\tau)) - \Lambda_0(t,\tau)\varphi_0(x,t,\tau) = 0 \quad (4.2.12)$$

$$(F_0(x,t,D_x,\tau))^*(\varphi_0(\cdot,t,\tau)) = 0 \quad (4.2.13)$$

$$\int \bar{\varphi}_0(x,t,\tau)\psi_0(x,t,\tau)dx = 1. \quad (4.2.14)$$

Here we denoted by $\varphi_0 \otimes \psi_0$ the operator $u = u(x) \mapsto \varphi_0 \int \bar{\psi}_0 u dx$; $\varphi_0 \otimes \psi_0$ must be a globally homogeneous symbol of degree zero.

Conditions (4.2.12) and (4.2.14) imply that $\psi_0 = \varphi_0$. Moreover (4.2.12) yields that

$$\Lambda_0(t,\tau) = \langle P_0(x,t,D_x,\tau)\varphi_0(x,t,\tau), \varphi_0(x,t,\tau)\rangle_{L^2(\mathbb{R}_x)},$$

coherently with the notation chosen above. Conditions (4.2.11) and (4.2.13) are rewritten as

$$\begin{aligned} P_0(x,t,D_x,\tau) \circ_x F_0(x,t,D_x,\tau) &= Id - \Pi_0 \\ F_0(x,t,D_x,\tau)(\varphi_0(\cdot,t,\tau)) &\in [\varphi_0]^\perp, \end{aligned}$$

whence

$$F_0(x,t,D_x,\tau) = \begin{cases} (P_0(x,t,D_x,\tau)_{|_{[\varphi_0]^\perp \cap B_q^2}})^{-1} & \text{on } [\varphi_0]^\perp \\ 0 & \text{on } [\varphi_0]. \end{cases} \quad (4.2.15)$$

Since P_0 is q-globally elliptic w.r.t. (x,ξ) smoothly depending on the parameters (t,τ), one can show that $F_0(x,t,D_x,\tau)$ is actually a pseudodifferential operator whose symbol verifies (4.1.10) with $m = k = -2$, $j = 0$, and is globally homogeneous of degree $-2/q$.

From now on we assume that $q < 2k$ and that $2k$ is not a multiple of q; the complementary cases are analogous.

Because of the fact that $P_{-j} = 0$ for $j = 1,\dots,q-1$, relations (4.2.11)–(4.2.14) are satisfied at degree $-j/q$, $j = 1,\dots,q-1$, by choosing $F_{-j} = 0$, $\psi_{-j} = 0$, $\Lambda_{-j} = 0$. Then we must examine homogeneity degree -1 in Equations (4.2.6)–(4.2.9). We get

$$\begin{aligned} P_{-q} \circ_x F_0 + P_0 \circ_x F_{-q} + \partial_\tau P_0 \circ_x D_t F_0 & \\ + \varphi_0 \otimes \psi_{-q} + \partial_\tau \varphi_0 \otimes D_t \varphi_0 &= 0 \quad (4.2.16) \\ P_0(\psi_{-q}) + P_{-q}(\varphi_0) + \partial_\tau P_0(D_t\varphi_0) & \\ - \Lambda_{-q}\varphi_0 - D_t\Lambda_0 \partial_\tau \varphi_0 &= 0 \quad (4.2.17) \\ (F_{-q})^*(\varphi_0) - (D_t F_0^*)(\partial_\tau \varphi_0) &= 0 \quad (4.2.18) \\ \langle \psi_{-q}, \varphi_0 \rangle_{L^2(\mathbb{R}_x)} + \langle D_t\varphi_0, \partial_\tau\varphi_0\rangle_{L^2(\mathbb{R}_x)} &= 0. \quad (4.2.19) \end{aligned}$$

First we solve w.r.t. $\psi_{-q} = \langle \psi_{-q}, \varphi_0 \rangle_{L^2(\mathbb{R}_x)} \varphi_0 + \psi_{-q}^{\perp} \in [\varphi_0] \oplus [\varphi_0]^{\perp}$. From (4.2.19) we get immediately that

$$\langle \psi_{-q}, \varphi_0 \rangle_{L^2(\mathbb{R}_x)} = -\langle D_t \varphi_0, \partial_\tau \varphi_0 \rangle_{L^2(\mathbb{R}_x)}. \tag{4.2.20}$$

(4.2.17) implies that

$$P_0(\langle \psi_{-q}, \varphi_0 \rangle \varphi_0) + P_0(\psi_{-q}^{\perp}) = -P_{-q}(\varphi_0) - \partial_\tau P_0(D_t \varphi_0) + \Lambda_{-q} \varphi_0 + D_t \Lambda_0 \partial_\tau \varphi_0.$$

Thus, using (4.2.20) we obtain that

$$\begin{aligned} [\varphi_0]^{\perp} \ni P_0(\psi_{-q}^{\perp}) \\ &= -P_{-q}(\varphi_0) - \partial_\tau P_0(D_t \varphi_0) + \Lambda_{-q} \varphi_0 + D_t \Lambda_0 \partial_\tau \varphi_0 + \langle D_t \varphi_0, \partial_\tau \varphi_0 \rangle \Lambda_0 \varphi_0, \end{aligned}$$

whence

$$\Lambda_{-q} = \langle P_{-q}(\varphi_0) + \partial_\tau P_0(D_t \varphi_0) - D_t \Lambda_0 \partial_\tau \varphi_0, \varphi_0 \rangle_{L^2(\mathbb{R}_x)} - \langle D_t \varphi_0, \partial_\tau \varphi_0 \rangle \Lambda_0. \tag{4.2.21}$$

$$\begin{aligned} \psi_{-q} = &-\langle D_t \varphi_0, \partial_\tau \varphi_0 \rangle_{L^2(\mathbb{R}_x)} \varphi_0 \\ &+ F_0 \left(-P_{-q}(\varphi_0) - \partial_\tau P_0(D_t \varphi_0) + D_t \Lambda_0 \partial_\tau \varphi_0 \right), \end{aligned} \tag{4.2.22}$$

since, by (4.2.15), $F_0 \varphi_0 = 0$. From (4.2.18) we deduce that, for every $u \in L^2(\mathbb{R}_x)$,

$$\Pi_0 F_{-q} u = \langle u, (D_t F_0^*)(\partial_\tau \varphi_0) \rangle_{L^2(\mathbb{R}_x)} \varphi_0 = [\varphi_0 \otimes (D_t F_0^*)(\partial_\tau \varphi_0)] \, u$$

Let $-\omega_{-q} = P_{-q} \circ_x F_0 + \partial_\tau P_0 \circ_x D_t F_0 + \varphi_0 \otimes \psi_{-q} + \partial_\tau \varphi_0 \otimes D_t \varphi_0$. Then from (4.2.15), applying F_0 to both sides of (4.2.16), we obtain that

$$(Id - \Pi_0) F_{-q} = -F_0 \omega_{-q}.$$

Therefore we deduce that

$$F_{-q} = \varphi_0 \otimes (D_t F_0^*)(\partial_\tau \varphi_0) - F_0 \omega_{-q}. \tag{4.2.23}$$

Inspecting (4.2.22), (4.2.23) we see that $\psi_{-q} \in H_q^{\frac{1}{2q}-1}$, globally homogeneous of degree $1/2q - 1$, $F_{-q} \in S_q^{-2,-2+\frac{q}{q-1}}$, globally homogeneous of degree $-2/q - 1$.

From (4.2.21) we have that $\Lambda_{-q} \in S_{1,0}^{2/q-1}$ homogeneous of degree $2/q - 1$. Moreover P_{-q} is $\mathscr{O}(t^{2\ell-1})$, $D_t \varphi_0$ is estimated by $t^{2\ell-1}$, for $t \to 0$, because of (4.2.2), $D_t \Lambda_0$ is also $\mathscr{O}(t^{2\ell-1})$ and $\Lambda_0 = \mathscr{O}(t^{2\ell})$ because of (4.2.3). We thus obtain that

$$\Lambda_{-q}(t, \tau) = \mathscr{O}(t^{2\ell-1}). \tag{4.2.24}$$

This ends the analysis of the terms of degree -1 in (4.2.5).

The procedure can be iterated arguing in a similar way. We would like to point out that the first homogeneity degree coming up and being not a negative integer is $-2k/q$ (we are availing ourselves of the fact that $2k$ is not a multiple of q. If it is a multiple of q, the above argument applies literally, but we need also the supplementary remark that we are going to make in the sequel.)

At homogeneity degree $-2k/q$ we do not see the derivatives w.r.t. t or τ of the symbols found at the previous levels, since they would only account for a negative integer homogeneity degrees.

In particular condition (4.2.7) for homogeneity degree $-2k/q$ reads as

$$P_0\psi_{-2k} + P_{-2k}\varphi_0 - \varphi_0\Lambda_{-2k} = 0.$$

Taking the scalar product of the above equation with the eigenfunction φ_0 and recalling that $\|\varphi_0(\cdot, t, \tau)\|_{L^2(\mathbb{R}^x)} = 1$, we obtain that

$$\Lambda_{-2k}(t, \tau) = \langle P_{-2k}\varphi_0, \varphi_0\rangle_{L^2(\mathbb{R}_x)} + \langle P_0\psi_{-2k}, \varphi_0\rangle_{L^2(\mathbb{R}_x)}. \tag{4.2.25}$$

Now, because of the structure of P_{-2k}, $\langle P_{-2k}\varphi_0, \varphi_0\rangle_{L^2(\mathbb{R}_x)} > 0$, while the second term on the right, which is equal to $\langle\psi_{-2k}, \varphi_0\rangle\bar{\Lambda}_0$, vanishes for $t = 0$. Thus if t is small enough we deduce that

$$\Lambda_{-2k}(t, \tau) > 0. \tag{4.2.26}$$

From this point on the procedure continues exactly as above.

We have thus proved the

Theorem 4.2.1. *The operator* Λ *defined in* (4.2.5) *is a pseudodifferential operator with symbol* $\Lambda(t, \tau) \in S^{2/q}_{1,0}(\mathbb{R}_t \times \mathbb{R}_\tau)$*. Moreover, if* j_0 *is a positive integer such that* $j_0 q < 2k < (j_0 + 1)q$*, the symbol of* Λ *has an asymptotic expansion of the form*

$$\Lambda(t, \tau) \sim \sum_{j=0}^{j_0} \Lambda_{-jq}(t, \tau) + \sum_{s\geq 0} \left(\Lambda_{-2k-sq}(t, \tau) + \Lambda_{-(j_0+1)q-sq}(t, \tau)\right). \tag{4.2.27}$$

Here Λ_{-p} *has homogeneity* $2/q - p/q$ *and*

a) $\Lambda_{-jq}(t, \tau) = \mathscr{O}(t^{2\ell-j})$ *for* $j = 0, \dots, j_0$.
b) Λ_{-2k} *satisfies* (4.2.26).

4.3. Hypoellipticity of P

In this section we give a different proof of the C^∞ hypoellipticity of P. This is accomplished by showing that the hypoellipticity of P follows from the hypoellipticity of Λ and proving that Λ is hypoelliptic if condition (1.2) is satisfied. As a matter of fact the hypoellipticity of P is equivalent to the hypoellipticity of Λ, so that the structure of Λ in Theorem 4.2.1, may be used to prove assertion (iii) in Theorem 1.1 (see [3].)

We state without proof the following

Lemma 4.3.1.

(a) *Let* $a \in S^{m,k}_q$*, properly supported, with* $k \leq 0$*. Then* $\operatorname{Op} a$ *is continuous from* $H^s_{\text{loc}}(\mathbb{R}^2)$ *to* $H^{s-m+k\frac{q-1}{q}}_{\text{loc}}(\mathbb{R}^2)$.

(b) *Let* $\varphi \in H^{m+\frac{1}{2q}}_q$*, properly supported. Then* $\operatorname{Op}\varphi$ *is continuous from* $H^s_{\text{loc}}(\mathbb{R})$ *to* $H^{s-m}_{\text{loc}}(\mathbb{R}^2)$*. Moreover* $\varphi^*(x, t, D_t)$ *is continuous from* $H^s_{\text{loc}}(\mathbb{R}^2)$ *to* $H^{s-m}_{\text{loc}}(\mathbb{R})$.

Mirroring the argument above, we can find symbols $F \in S_q^{-2,-2}$, $\psi \in H_q^{1/2q}$ and $\Lambda \in S_{1,0}^{2/q}$ as in (4.2.10), such that

$$\begin{bmatrix} F(x,t,D_x,D_t) & \psi(x,t,D_t) \\ \psi^*(x,t,D_t) & -\Lambda(t,D_t) \end{bmatrix} \circ \begin{bmatrix} P(x,t,D_x,D_t) & \varphi_0(x,t,D_t) \\ \varphi_0^*(x,t,D_t) & 0 \end{bmatrix} \equiv \begin{bmatrix} Id_{C_0^\infty(\mathbb{R}^2)} & 0 \\ 0 & Id_{C_0^\infty(\mathbb{R})} \end{bmatrix}. \quad (4.3.1)$$

From (4.3.1) we get the couple of relations

$$F(x,t,D_x,D_t) \circ P(x,t,D_x,D_t) = Id - \psi(x,t,D_t) \circ \varphi_0^*(x,t,D_t) \quad (4.3.2)$$

$$\psi^*(x,t,D_t) \circ P(x,t,D_x,D_t) = \Lambda(t,D_t) \circ \varphi_0^*(x,t,D_t). \quad (4.3.3)$$

Proposition 4.3.1. *If Λ is hypoelliptic with a loss of δ derivatives, then P is also hypoelliptic with a loss of derivatives equal to*

$$2\frac{q-1}{q} + \max\{0, \delta\}.$$

Proof. Assume that $Pu \in H^s_{\text{loc}}(\mathbb{R}^2)$. From Lemma 4.3.1 we have that $FPu \in H^{s+2/q}_{\text{loc}}(\mathbb{R}^2)$. By (4.3.2) we have that $u - \psi\varphi_0^* u \in H^{s+2/q}_{\text{loc}}(\mathbb{R}^2)$. Again, using Lemma 4.3.1, $\psi^* Pu \in H^s_{\text{loc}}(\mathbb{R})$, so that, by (4.3.3), $\Lambda\varphi_0^* u \in H^s_{\text{loc}}(\mathbb{R})$. The hypoellipticity of Λ yields then that $\varphi_0^* u \in H^{s+\frac{2}{q}-\delta}_{\text{loc}}(\mathbb{R})$. From Lemma 4.3.1 we obtain that $\psi\varphi_0^* u \in H^{s+\frac{2}{q}-\delta}_{\text{loc}}(\mathbb{R})$. Thus $u = (Id - \psi\varphi_0^*)u + \psi\varphi_0^* u \in H^{s+\frac{2}{q}-\max\{0,\delta\}}_{\text{loc}}$. This proves the proposition. □

Next we prove the hypoellipticity of Λ under the assumption that $\ell > k/q$.

First we want to show that there exists a smooth non negative function $M(t,\tau)$, such that

$$M(t,\tau) \le C|\Lambda(t,\tau)|, \qquad |\Lambda^{(\alpha)}_{(\beta)}(t,\tau)| \le C_{\alpha,\beta} M(t,\tau)(1+|\tau|)^{-\rho\alpha+\delta\beta}, \quad (4.3.4)$$

where α, β are non negative integers, C, $C_{\alpha,\beta}$ suitable positive constants and the inequality holds for t in a compact neighborhood of the origin and $|\tau|$ large. Moreover ρ and δ are such that $0 \le \delta < \rho \le 1$.

We actually need to check the above estimates for Λ only when τ is positive and large.

Let us choose $\rho = 1$, $\delta = \frac{k}{\ell q} < 1$ and

$$M(t,\tau) = \tau^{\frac{2}{q}} \left(t^{2\ell} + \tau^{-\frac{2k}{q}} \right),$$

for $\tau \ge c \ge 1$. It is then evident, from Theorem 4.2.1, that the first of the conditions (4.3.4) is satisfied. The second condition in (4.3.4) is also straightforward for $\Lambda_0 + \Lambda_{-2k}$, because of (4.2.26) and (4.2.3). To verify the second condition in (4.3.4) for Λ_{-jq}, $q \in \{1, \dots, j_0\}$, we have to use property a- in the statement of Theorem 4.2.1. Finally the verification is straightforward for the lower order parts of the symbol in Formula (4.2.27). Using Theorem 22.1.3 of [10], we see that there exists a parametrix for Λ. Moreover from the proof of the above-quoted theorem we get that the symbol of any parametrix satisfies the same estimates that Λ^{-1} satisfies, i.e.,

$$|D_t^\beta D_\tau^\alpha \Lambda(t,\tau)| \le C_{\alpha,\beta}\left[\tau^{\frac{2}{q}}\left(t^{2\ell}+\tau^{-\frac{2k}{q}}\right)\right]^{-1}(1+\tau)^{-\alpha+\frac{k}{\ell q}\beta}$$
$$\le C_{\alpha,\beta}(1+\tau)^{\frac{2k}{q}-\frac{2}{q}-\alpha+\frac{k}{\ell q}\beta},$$

for t in a compact set and $\tau \ge C$. Thus the parametrix obtained from Theorem 22.1.3 of [10] has a symbol in $S^{\frac{2k}{q}-\frac{2}{q}}_{1,\frac{k}{\ell q}}$.

We may now state the

Theorem 4.3.1. *Λ is hypoelliptic with a loss of $\frac{2k}{q}$ derivatives, i.e., $\Lambda u \in H^s_{\text{loc}}$ implies that $u \in H^{s+\frac{2}{q}-\frac{2k}{q}}_{\text{loc}}$.*

Theorem 4.3.1 together with Proposition 4.3.1 prove assertion (i) of Theorem 1.1.

A. Appendix

We prove here a well-known formula for the adjoint of a product of two pseudodifferential operators using just symbolic calculus. Let a, b symbols in $S^0_{1,0}(\mathbb{R}_t)$. We want to show that

$$(a\#b)^* = b^*\#a^*, \tag{A.1}$$

where $\#$ denotes the usual symbolic composition law (a higher-dimensional extension involves just a more cumbersome notation.)

We may write

$$\begin{aligned}(a\#b)^* &= \sum_{\ell,\alpha\ge 0}\frac{(-1)^\alpha}{\alpha!\ell!}\partial_\tau^\ell D_t^\ell\left(\partial_\tau^\alpha \bar{a} D_t^\alpha \bar{b}\right)\\ &= \sum_{\ell,\alpha\ge 0}\sum_{r,s\le \ell}\frac{(-1)^\alpha}{\alpha!\ell!}\binom{\ell}{r}\binom{\ell}{s}\partial_\tau^{\alpha+r}D_t^{\ell-s}\bar{a}\;\partial_\tau^{\ell-r}D_t^{\alpha+s}\bar{b}.\end{aligned}$$

Let us change the summation indices according to the following prescription; $j = \alpha + r$, $\beta + j = \ell - s$, $i = \alpha + s$, so that $\ell - r = i + \beta$, we may rewrite the last equality in the above formula as

$$(a\#b)^*$$
$$= \sum_{i,j,\beta\ge 0}\sum_{s\le i}\frac{(-1)^{i-s}}{(i-s)!(\beta+j+s)!}\binom{\beta+j+s}{j-i+s}\binom{\beta+j+s}{s}\partial_\tau^{i+\beta}D_t^i\bar{b}\;\partial_\tau^j D_t^{\beta+j}\bar{a}.$$

Let us examine the s-summation; we claim that

$$\sum_{s=0}^{i}\frac{(-1)^{i-s}}{(i-s)!}\frac{1}{(\beta+i)!(j-i+s)!}\binom{\beta+j=s}{s} = \frac{1}{\beta!i!j!}.$$

This is actually equivalent to

$$\sum_{s=0}^{i}(-1)^{i-s}\binom{i}{s}\binom{\beta+j+s}{\beta+i}=\binom{\beta+j}{j}.$$

Setting $i-s=\nu\in\{0,1,\dots,i\}$, the above relation is written as

$$\sum_{\nu=0}^{i}(-1)^{\nu}\binom{i}{\nu}\binom{\beta+i+j-\nu}{\beta+i}=\binom{\beta+j}{j},$$

and this is precisely identity (12.15) in W. Feller [8], vol. 1.

Thus we may conclude that

$$\begin{aligned}(a\#b)^* &= \sum_{i,j,\beta}\frac{1}{\beta!i!j!}\partial_\tau^{i+\beta}D_t^i\bar{b}\partial_\tau^j D_t^{j+\beta}\bar{a}\\ &= \sum_{\beta\geq 0}\frac{1}{\beta!}\partial_\tau^\beta\left(\sum_{i\geq 0}\frac{1}{i!}\partial_\tau^i D_t^i\bar{b}\right)D_t^\beta\left(\sum_{j\geq 0}\frac{1}{j!}\partial_\tau^j D_t^j\bar{a}\right)\\ &= b^*\#a^*.\end{aligned}$$

This proves (A.1).

As a by-product of the above argument we get the following identity

$$\sum_{i,j,\beta}\frac{1}{\beta!i!j!}\partial_\tau^{i+\beta}D_t^i\bar{b}\partial_\tau^j D_t^{j+\beta}\bar{a}=\sum_{\ell,\alpha\geq 0}\frac{(-1)^\alpha}{\alpha!\ell!}\partial_\tau^\ell D_t^\ell\left(\partial_\tau^\alpha\bar{a}D_t^\alpha\bar{b}\right), \tag{A.2}$$

which is the purpose of the present Appendix.

We would like to point out that the relation $(a^*)^*=a$ rests on the identity

$$\begin{aligned}&\sum_{\ell\geq 0}\frac{1}{\ell!}\partial_\tau^\ell D_t^\ell\overline{\left(\sum_{\alpha\geq 0}\frac{1}{\alpha!}\partial_\tau^\alpha D_t^\alpha\bar{a}\right)}\\ &\qquad=\sum_{s\geq 0}\frac{1}{s!}\left(\sum_{\ell+\alpha=s}\frac{s!}{\ell!\alpha!}(-1)^\alpha\right)\partial_\tau^s D_t^s a=\sum_{s\geq 0}\frac{1}{s!}(1-1)^s\partial_\tau^s D_t^s a=a.\end{aligned} \tag{A.3}$$

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Subelliptic Estimates

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Dedicated to Linda Rothschild

Abstract. This paper gathers old and new information about subelliptic estimates for the $\overline{\partial}$-Neumann problem on smoothly bounded pseudoconvex domains. It discusses the failure of effectiveness of Kohn's algorithm, gives an algorithm for triangular systems, and includes some new information on sharp subelliptic estimates.

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1. Introduction

The purpose of this paper is to clarify some issues concerning subelliptic estimates for the $\overline{\partial}$-Neumann problem on $(0, 1)$ forms. Details of several of the results and examples here do not appear in the literature, but versions of them have been known to the authors and a few others for a long time, and some have been mentioned without proof such as in [DK]. Recent interest in this subject helps justify including them. Furthermore, the situation in two complex dimensions has long been completely understood; one of the main results there is due to Rothschild and Stein ([RS]) and hence fits nicely into this volume.

First we briefly recall the definition of subelliptic estimate and one consequence of such an estimate. See [BS], [C1], [C2], [C3], [DK], [K4], [K5], [KN] for considerable additional discussion. We then discuss the situation in two complex dimensions, where things are completely understood. We go on to describe two methods for proving such estimates, Kohn's method of subelliptic multipliers and Catlin's method of construction of bounded plurisubharmonic functions with large Hessians.

We provide in Proposition 4.4 an example exhibiting the failure of effectiveness for Kohn's algorithm for finding subelliptic multipliers, and we give a simplified situation (Theorem 5.1) in which one can understand this algorithm perfectly. This section is taken from [D5]. We go on to discuss some unpublished examples of the first author. These examples provide surprising but explicit information about how the largest possible value of the parameter ϵ that arises in a subelliptic estimate is related to the geometry of the boundary. See Example 7.1 and Theorem 7.2.

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2. Definition of subelliptic estimates

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, and assume that $p \in b\Omega$. Let $T^{1,0}b\Omega$ be the bundle whose sections are $(1,0)$ vectors tangent to $b\Omega$. We may suppose that there is a neighborhood of p on which $b\Omega$ is given by the vanishing of a smooth function r with $dr(p) \neq 0$. In coordinates, a vector field $L = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}$ is a local section of $T^{1,0}b\Omega$ if, on $b\Omega$

$$\sum_{j=1}^{n} a_j(z) r_{z_j}(z) = 0. \tag{1}$$

Then $b\Omega$ is pseudoconvex at p if, whenever (1) holds we have

$$\sum_{j,k=1}^{n} r_{z_j \overline{z}_k}(p) a_j(p) \overline{a_k(p)} \geq 0. \tag{2}$$

It is standard to express (2) more invariantly. The bundle $T^{1,0}(b\Omega)$ is a subbundle of $T(b\Omega) \otimes \mathbb{C}$. The intersection of $T^{1,0}(b\Omega)$ with its complex conjugate bundle is the zero bundle, and their direct sum has fibers of codimension one in $T(b\Omega) \otimes \mathbb{C}$. Let η be a non-vanishing purely imaginary 1-form that annihilates this direct sum. Then (1) and (2) together become

$$\lambda(L, \overline{L}) = \langle \eta, [L, \overline{L}] \rangle \geq 0 \tag{3}$$

on $b\Omega$ for all local sections of $T^{1,0}(b\Omega)$. Formula (3) defines a Hermitian form λ on $T^{1,0}(b\Omega)$ called the Levi form. The Levi form is defined only up to a multiple, but this ambiguity makes no difference in what we will do. The domain Ω or its boundary $b\Omega$ is called *pseudoconvex* if the Levi form is definite everywhere on $b\Omega$; in this case, we multiply by a constant to ensure that it is *nonnegative* definite. The boundary is *strongly pseudoconvex* at p if the Levi form is positive definite there. Each smoothly bounded domain has an open subset of strongly pseudoconvex boundary points; the point farthest from the origin must be strongly pseudoconvex, and strong pseudoconvexity is an *open condition*.

Subelliptic estimates arise from considering the $\overline{\partial}$-complex on the closed domain $\overline{\Omega}$. As usual in complex geometry we have notions of smooth differential forms of type (p, q). We will be concerned only with the case of $(0,1)$ forms here; similar examples and results apply for forms of type (p, q).

A smooth differential $(0,1)$ form $\sum_{j=1}^{n} \phi_j d\overline{z}^j$, defined near p, lies in the domain of $\overline{\partial}^*$ if the vector field $\sum_{j=1}^{n} \phi_j \frac{\partial}{\partial z_j}$ lies in $T_z^{1,0} b\Omega$ for z near p. The boundary condition for being in the domain of $\overline{\partial}^*$ therefore becomes $\sum \phi_j \frac{\partial r}{\partial z_j} = 0$ on the set where $r = 0$. Let $\|\psi\|$ denote the L^2 norm and let $\|\psi\|_\epsilon$ denote the Sobolev ϵ norm of ψ, where ψ can be either a function or a differential form. The Sobolev norm involves fractional derivatives of order ϵ of the components of ψ.

Definition 2.1. A subelliptic estimate holds on $(0,1)$ forms at p if there is a neighborhood U of p and positive constants C and ϵ such that (4) holds for all forms ϕ, compactly supported in U and in the domain of $\overline{\partial}^*$.

$$\|\phi\|_\epsilon^2 \leq C \left(\|\overline{\partial}\phi\|^2 + \|\overline{\partial}^*\phi\|^2 + \|\phi\|^2 \right). \tag{4}$$

In this paper we relate the largest possible value of the parameter ϵ for which (4) holds to the geometry of $b\Omega$.

Perhaps the main interest in subelliptic estimates is the fundamental local regularity theorem of Kohn and Nirenberg [KN]. In the statement of the theorem, the canonical solution to the inhomogeneous Cauchy-Riemann equation is the unique solution orthogonal to the holomorphic functions.

Theorem 2.1. *Let Ω be a smoothly bounded pseudoconvex domain, and assume that there is a subelliptic estimate at a boundary point p. Then there is a neighborhood U of p in $\overline{\Omega}$ with the following property. Let α be a $(0,1)$ form with L^2 coefficients and $\overline{\partial}\alpha = 0$. Let u be the canonical solution to $\overline{\partial}u = \alpha$. Then u is smooth on any open subset of U on which α is smooth.*

It has been known for nearly fifty years ([K1], [K2], [FK]) that there is a subelliptic estimate with $\epsilon = \frac{1}{2}$ at each strongly pseudoconvex boundary point. One is also interested in *global* regularity. See [BS] for a survey of results on global regularity of the canonical solution. In particular, on each smoothly bounded pseudoconvex domain, there is a smooth solution to $\overline{\partial}u = \alpha$ when α is smooth and $\overline{\partial}\alpha = 0$, but the canonical solution itself need not be smooth.

3. Subelliptic estimates in two dimensions

Let Ω be a pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary M, and suppose $p \in M$. The statement of Theorem 3.1 below, resulting by combining the work of several authors, completely explains the situation.

Assume that r is a defining function for M near p. We may choose coordinates such that p is the origin and

$$r(z) = 2\mathrm{Re}(z_2) + f(z_1, \mathrm{Im}(z_2)), \tag{5}$$

where $df(0) = 0$. We let $\mathbf{T}(M,p)$ denote the maximum order of contact of one-dimensional complex analytic curves with M at p, and we let $\mathbf{T}_{reg}(M,p)$ denote the maximum order of contact of one-dimensional regular complex analytic curves with M at p. We let $t(M,p)$ denote the *type* of M at p, defined as follows. Let L be a type $(1,0)$ vector field on M, with $L(p) \neq 0$. Then *type*(L,p) is the smallest integer k such that there is an iterated bracket $\mathcal{L}_k = [\dots[L_1, L_2], \dots, L_k]$ for which each L_j is either L or $\overline{L}$ and such that

$$\langle \mathcal{L}_k, \eta \rangle(p) \neq 0.$$

This number measures the degeneracy of the Levi form at p. It is independent of the choice of L, as $T_p^{1,0}M$ is one-dimensional. We put $t(M,p) = type(L,p)$.

In two dimensions there is an equivalent method for computing $t(M,p)$. Consider the Levi form $\lambda(L, \overline{L})$ as a function defined near p. We ask how many derivatives one must take in either the L or $\overline{L}$ direction to obtain something non-zero at p. Then $c(L,p)$ is defined to be two more than this minimum number of derivatives; we add two because the Levi form already involves two derivatives. In two dimensions it is easy to see that *type*$(L,p) = c(L,p)$. This conclusion is false in higher dimensions when the Levi form has eigenvalues of opposite signs at p. It is likely to be true on pseudoconvex domains; see [D1] for more information.

In $\mathbb{C}^2$ there are many other ways to compute the type of a point. The easiest one involves looking at the defining function directly. With f as in (5), both of these concepts and also both versions of orders of contact mentioned above equal the order of vanishing of the function $f(z_1, 0)$ at the origin. Things are much more subtle and interesting in higher dimensions regarding these various measurements. See [D1]. Both the geometry and the estimates are easier in $\mathbb{C}^2$ than in higher dimensions; the following theorem explains fully the two-dimensional case.

Theorem 3.1. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$, and suppose $p \in b\Omega$. The following are equivalent:*

1) *There is a subelliptic estimate at p with $\epsilon = \frac{1}{2m}$, but for no larger value of ϵ.*
2) *For L a $(1,0)$ vector field on $b\Omega$ with $L(p) \neq 0$, we have type$(L,p) = 2m$.*
3) *For L as in 2), we have $c(L,p) = 2m$.*
4) *There is an even integer $2m$ such that $\mathbf{T}(b\Omega, p) = 2m$.*
5) *There is an even integer $2m$ such that $\mathbf{T}_{\text{reg}}(M,p) = 2m$.*

Kohn [K3] established the first subelliptic estimate for domains in $\mathbb{C}^2$, assuming that *type*(L,p) was finite. Greiner [Gr] established the converse. To establish the sharp result that ϵ could be chosen to be the reciprocal of *type*(L,p), Kohn invoked results of Rothschild-Stein [RS] based on the notion of nilpotent Lie groups. These difficult results establish the equivalence of 1) and 2) above. Also see for example [CNS] among the many references for estimates in other function spaces for solving the Cauchy-Riemann equations in two dimensions.

The geometry in two dimensions is easy to understand; it is quite easy to establish that condition 2) is equivalent to the other conditions from Theorem 3.1, and hence we listed all five conditions. In higher dimensions, however, the geometry is completely different. Nonetheless, based on Theorem 3.1, one naturally seeks a geometric condition for subellipticity in higher dimensions.

4. Subelliptic multipliers

We next consider the approach of Kohn from [K4] for proving subelliptic estimates. Let $\mathcal{E}$ denote the ring of germs of smooth functions at p. Recall that $\|u\|$ denotes the L^2-norm of u; we use this notation whether u is a function, a 1-form, or a 2-form. We write $\|u\|_\epsilon$ for the Sobolev ϵ norm.

Definition 4.1. Assume $f \in \mathcal{E}$. We say that f is a subelliptic multiplier at p if there are positive constants C and ϵ and a neighborhood U such that

$$\|f\phi\|_\epsilon^2 \leq C\left(\|\overline{\partial}\phi\|^2 + \|\overline{\partial}^*\phi\|^2 + \|\phi\|^2\right) \tag{6}$$

for all forms ϕ supported in U and in the domain of $\overline{\partial}^*$.

We will henceforth write $Q(\phi, \phi)$ for $\|\overline{\partial}\phi\|^2 + \|\overline{\partial}^*\phi\|^2 + \|\phi\|^2$. By Definitions 2.1 and 4.1, a subelliptic estimate holds at p if and only if the constant function 1 is a subelliptic multiplier at p. We recall that when $b\Omega$ is strongly pseudoconvex at p we can take $\epsilon = \frac{1}{2}$ in (4).

The collection of subelliptic multipliers is a non-trivial ideal in $\mathcal{E}$ closed under taking radicals. Furthermore, the defining function r and the determinant of the Levi form $\det(\lambda)$ are subelliptic multipliers. We state these results of Kohn [K4]:

Proposition 4.1. *The collection I of subelliptic multipliers is a radical ideal in $\mathcal{E}$; in particular, if $f^N \in I$ for some N, then $f \in I$. Also, r and $\det(\lambda)$ are in I.*

$$\|r\phi\|_1^2 \leq CQ(\phi, \phi) \tag{7}$$

$$\|\det(\lambda)\phi\|_{\frac{1}{2}}^2 \leq CQ(\phi, \phi). \tag{8}$$

Kohn's algorithm starts with these two subelliptic multipliers and constructs additional ones. We approach the process via the concept of *allowable rows.* An n-tuple $(f_1, \ldots, f_n)$ of germs of functions is an allowable row if there are positive constants C and ϵ such that, for all ϕ as in the definition of subelliptic estimate,

$$\left\|\sum_j f_j\phi_j\right\|_\epsilon^2 \leq CQ(\phi, \phi). \tag{9}$$

The most important example of allowable row is, for each j, the jth row of the Levi form, namely the n-tuple $(r_{z_1\overline{z}_j}, \ldots, r_{z_n\overline{z}_j})$.

The following fundamental result of Kohn enables us to pass between allowable rows and subelliptic multipliers:

Proposition 4.2. *Let f be a subelliptic multiplier such that*

$$\|f\phi\|_{2\epsilon}^2 \leq Q(\phi, \phi). \tag{10}$$

Then the n-tuple of functions $(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_1})$ is an allowable row, and we have:

$$\left\|\sum_j \frac{\partial f}{\partial z_j}\phi_j\right\|_\epsilon^2 \leq CQ(\phi, \phi). \tag{11}$$

Conversely, consider any $n \times n$ matrix (f_{ij}) of allowable rows. Then $\det(f_{ij})$ is a subelliptic multiplier.

Proof. See [K4] or [D1]. □

For domains with real analytic boundary, Kohn's process always terminates in finitely many steps, depending on only the dimension. Following the process produces two lists of finite length; one of modules of allowable rows, the other of subelliptic multipliers. The value of the ϵ obtained from this process depends on both the length of this list and the number of radicals taken in each step. We will show that there is *no positive lower bound* on the value of ϵ in a subelliptic estimate obtained from Kohn's process in general. In order to do so we recall some geometric information and notation from [D1] and [D2].

For a real hypersurface M in $\mathbb{C}^n$, we recall that $\mathbf{T}(M,p)$ denotes the maximum order of contact of one-dimensional complex analytic varieties with M at p. We compute this number as follows. Let $\nu(z)$ denote the order of vanishing operator. Let z be a parametrized holomorphic curve with $z(0) = p$. We compute the ratio $\mathbf{T}(M,p,z) = \frac{\nu(z^*r)}{\nu(z)}$ and call it the order of contact of the curve z with M at p. Then $\mathbf{T}(M,p)$ is the supremum over z of $\mathbf{T}(M,p,z)$. Later we will generalize this concept.

Next we consider the ring of germs of holomorphic functions $\mathcal{O}$ at 0 in $\mathbb{C}^n$. Some of the ideas also apply to the formal power series ring; at times we write R or R_n when the statement applies in either setting. See [Cho] for a treatment of Kohn's algorithm in the formal power series setting.

The maximal ideal in $\mathcal{O}$ is denoted by $\mathbf{m}$. If I is a proper ideal in $\mathcal{O}$, then the Nullstellensatz guarantees that its variety $\mathbf{V}(I)$ is an isolated point if and only if the radical of I equals $\mathbf{m}$. In this case the intersection number $\mathbf{D}(I)$ plays an important role in our discussions. We put

$$\mathbf{D}(I) = \dim_{\mathbb{C}} \mathcal{O}/I.$$

For such an ideal I we also consider its order of contact $\mathbf{T}(I)$, defined analogously to the order of contact with a hypersurface. This number provides a slightly different measurement of the singularity than does $\mathbf{D}(I)$. See [D1] and [D5] for precise information.

The following proposition is a special case of results from [D2] and [K4]. It gives a simple situation where one can relate the geometry to the estimates. Note that the geometric conditions 3) through 6) state in various ways that there is no complex analytic curve in $b\Omega$ through 0.

Proposition 4.3. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ for which $0 \in b\Omega$, and there are holomorphic functions h_j such that the defining equation near 0 can be written as*

$$r(z) = \mathrm{Re}(z_n) + \sum_{j=1}^{N} |h_j(z)|^2. \tag{12}$$

The following are equivalent:

1) *There is a subelliptic estimate on $(0,1)$ forms.*

2) *There is no complex analytic* (*one-dimensional*) *curve passing through* 0 *and lying in* $b\Omega$.
3) $\mathbf{T}(b\Omega, 0)$ *is finite.*
4) $\mathbf{V}(z_n, h_1, \ldots, h_N) = \{0\}$.
5) *The radical of the ideal* $(z_n, h_1, \ldots, h_N)$ *is* $\mathbf{m}$.
6) $\mathbf{D}(z_n, h_1, \ldots, h_N)$ *is finite.*

Our next example is of the form (12), but it illustrates a new quantitative result. Let Ω be a pseudoconvex domain in $\mathbb{C}^3$ whose defining equation near the origin is given by

$$r(z) = \mathrm{Re}(z_3) + |z_1^M|^2 + |z_2^N + z_2 z_1^K|^2. \tag{13}$$

We assume that $K > M \geq 2$ and $N \geq 3$. We note that $\mathbf{T}(b\Omega, 0) = 2\max(M, N)$ and that $\mathbf{D}(z_1^M, z_2^N + z_2 z_1^K, z_3) = MN$. In the next result we show that Kohn's algorithm for finding subelliptic multipliers gives no lower bound for ϵ in terms of the dimension and the type.

Proposition 4.4 (Failure of effectiveness). *Let* Ω *be a pseudoconvex domain whose boundary contains* 0, *and which is defined near* 0 *by* (13). *Then the root taken in the radical required in the second step of Kohn's algorithm for subelliptic multipliers is at least* K, *and hence it is independent of the type at* 0. *In particular, the procedure in* [K4] *gives no positive lower bound for* ϵ *in terms of the type.*

Proof. Let Ω be a domain in $\mathbb{C}^{n+1}$ defined near the origin by (13). By the discussion in [K4], [D1] or [D5], Kohn's algorithm reduces to an algorithm in the ring $\mathcal{O}$ in two dimensions. We therefore write the variables as (z, w) and consider the ideal (h) defined by $(z^M, w^N + wz^K)$ in two variables. The exponents are positive integers; we assume $K > M \geq 2$ and $N \geq 3$. Note that $\mathbf{D}(h) = MN$ and $\mathbf{T}(h) = \max(M, N)$. We write $g(z, w) = w^N + wz^K$ and we use subscripts on g to denote partial derivatives.

The algorithm begins with the collection $\mathcal{M}_0$ of *allowable rows* spanned by (14) and the ideal I_0 given in (15):

$$\begin{pmatrix} z^{M-1} & 0 \\ g_z & g_w \end{pmatrix} \tag{14}$$

There is only one determinant to take, and therefore

$$I_0 = \mathrm{rad}(z^{M-1} g_w) = (z g_w). \tag{15}$$

By definition $\mathcal{M}_1$ is the union of $\mathcal{M}_0$ and $d(zg_w) = (zg_{wz} + g_w)dz + zg_{ww}dw$. Using the row notation as before we see that the spanning rows of $\mathcal{M}_1$ are given by (16):

$$\begin{pmatrix} z^{M-1} & 0 \\ g_z & g_w \\ zg_{wz} + g_w & zg_{ww} \end{pmatrix}. \tag{16}$$

It follows that I_1 is the radical of the ideal J_1 generated by the three possible determinants.

The ideal generated by zg_w and the two new determinants is

$$J_1 = (zg_w, z^M g_{ww}, zg_z g_{ww} - zg_w g_{zw} - g_w^2). \tag{17}$$

It is easy to see that

$$I_1 = \mathrm{rad}(J_1) = \mathbf{m}. \tag{18}$$

Thus $\mathcal{M}_2$ includes dz and dw and hence $I_2 = (1)$.

The crucial point concerning effectiveness involves the radical taken in passing from J_1 to I_1. We prove that we cannot bound this root in terms of M and N. To verify this statement we claim that z^{K-1} is not an element of J_1. This claim shows that the number of roots taken must be at least K. Since K can be chosen independently of M and N and also arbitrarily large, there is no bound on the number of roots taken in terms of the dimension 2 and the intersection number $\mathbf{D}(h) = MN$ or the order of contact $\mathbf{T}(I) = \max(M, N)$.

It remains to prove the claim. If $z^{K-1} \in J_1$, then we could write

$$z^{K-1} = a(z,w)zg_w + b(z,w)z^M g_{ww} + c(z,w)(zg_z g_{ww} - zg_{zw} - g_w^2) \tag{19}$$

for some a, b, c. We note that $g_{ww}(z,0) = 0$, that $g_w(z,0) = z^K$, and $g_{zw}(z,0) = Kz^{K-1}$. Using this information we set $w = 0$ in (19) and obtain

$$z^{K-1} = a(z,0)z^K + b(z,0)0 + c(z,0)(-zKz^{K-1} + 0). \tag{20}$$

It follows from (20) that z^{K-1} is divisible by z^K; this contradiction proves that z^{K-1} is not in J_1, and hence that passing to I_1 requires at least K roots. (It is easy to show, but the information is not needed here, that taking K roots suffices.) □

This proposition shows that one cannot take radicals in a controlled fashion unless one revises the algorithm. One might naturally ask whether we can completely avoid taking radicals. The following example shows otherwise.

Example 4.1. Put $n = 2$, and let h denote the three functions (z^2, zw, w^2). Then the three Jacobians obtained are $(z^2, 2w^2, 4zw)$. If we tried to use the ideal generated by them, instead of its radical, then the algorithm would get stuck. We elaborate; the functions z^2, zw, w^2 are not known to be subelliptic multipliers at the start. After we compute I_0, however, they are known to be subelliptic multipliers and hence we are then allowed to take the radical. This strange phenomenon (we cannot use these functions at the start, but we can use them after one step) illustrates one of the subtleties in Kohn's algorithm.

5. Triangular systems

Two computational difficulties in Kohn's algorithm are finding determinants and determining radicals of ideals. We describe a nontrivial class of examples for which finding the determinants is easy. At each stage we require only determinants of triangular matrices. Furthermore we avoid the computation of uncontrolled radicals; for this class of examples we never take a root of order larger than the underlying dimension. In order to do so, we deviate from Kohn's algorithm by treating the modules of $(1,0)$ forms differently.

We call this class of examples *triangular systems*. The author introduced a version of these examples in [D4], using the term *regular coordinate domains*, but the calculations there give a far from optimal value of the parameter ϵ in a subelliptic estimate. The version in this section thus improves the work from [D4]. Catlin and Cho [CC] and independently Kranh and Zampieri [KZ] have recently established subelliptic estimates in some specific triangular systems. The crucial point in this section is that triangular systems enable one to choose allowable rows in Kohn's algorithm, one at a time and with control on all radicals. In Theorem 5.1 we establish a decisive result on effectiveness for triangular systems.

Definition 5.1 (Triangular Systems). Let $\mathcal{H}$ be a collection of nonzero elements of $\mathbf{m} \subset R_n$. We say that $\mathcal{H}$ is a *triangular system of full rank* if, possibly after a linear change of coordinates, there are elements, $h_1, \ldots, h_n \in \mathcal{H}$ such that

1) For each i with $1 \leq i \leq n$, we have $\frac{\partial h_i}{\partial z_j} = 0$ whenever $j > i$. In other words, h_i depends on only the variables $z_1, \ldots, z_i$.
2) For each i with $1 \leq i \leq n$, $h_i(0, z_i) \neq 0$. Here $(0, z_i)$ is the i-tuple $(0, \ldots, 0, z_i)$.

It follows from 1) that the derivative matrix $dh = (\frac{\partial h_i}{\partial z_j})$ for $1 \leq i, j \leq n$ is lower triangular. (All the entries above the main diagonal vanish identically.) It follows from 2) that $\frac{\partial h_i}{\partial z_i}(0, z_i) \neq 0$. By combining these facts we see that $J = \det(dh)$ is not identically zero. Our procedure makes no use of the other elements of $\mathcal{H}$.

Of course any ideal defining a zero-dimensional variety contains a triangular system of full rank. We are assuming here additionally that the differentials of these functions define the initial module of allowable rows.

Remark 5.1. Triangular systems of rank less than n are useful for understanding the generalization of the algorithm where we consider q by q minors. We do not consider these systems here, and henceforth we drop the phrase *of full rank*, assuming that our triangular systems have full rank.

Let $\mathcal{H}$ be a triangular system. After renumbering, we may assume that h_1 is a function of z_1 alone, h_2 is a function of (z_1, z_2), and so on. Note that $h_1(z_1) = z_1^{m_1} u_1(z_1)$ for a unit u_1, that $h_2(z_1, z_2) = z_2 u_2(z_2) + z_1 g_2(z_1, z_2)$ for a unit u_2, and so on. After changing coordinates again we may assume that these units are constant. For example $z_1^{m_1} u_1(z_1) = \zeta_1^{m_1}$, where ζ_1 is a new coordinate. We may therefore assume that a triangular system includes functions $h_1, \ldots, h_n$ as follows:

$$h_1(z) = z_1^{m_1} \tag{21.1}$$

$$h_2(z) = z_2^{m_2} + z_1 g_{21}(z_1, z_2) \tag{21.2}$$

$$h_3(z) = z_3^{m_3} + z_1 g_{31}(z_1, z_2, z_3) + z_2 g_{32}(z_1, z_2, z_3) \tag{21.3}$$

$$h_n(z) = z_n^{m_n} + \sum_{j=1}^{n-1} z_j g_{nj}(z_1, \ldots, z_n). \tag{21.n}$$

In (21) the holomorphic germs g_{kl} are arbitrary. Our approach works uniformly in them (Corollary 5.1), but the ϵ from Kohn's algorithm depends upon them.

Each h_j depends upon only the first j variables and has a pure monomial in z_j. A useful special case is where each h_j is a Weierstrass polynomial of degree m_j in z_j whose coefficients depend upon only the first $j-1$ variables.

Example 5.1. Write the variables (z, w) in two dimensions. The pair of functions

$$h(z,w) = (h_1(z,w), h_2(z,w)) = (z^m, w^n + zg(z,w)), \tag{22}$$

where g is any element of R_2, form a triangular system.

Lemma 5.1. *Let $h_1, \ldots, h_n$ define a triangular system in R_n and let (h) denote the ideal generated by them. Then*

$$\mathbf{D}(h) = \prod_{j=1}^{n} m_j. \tag{23}$$

Proof. There are many possible proofs. One is to compute the vector space dimension of $R_n/(h)$ by listing a basis of this algebra. The collection $\{z^\alpha\}$ for $0 \le \alpha_i \le m_i - 1$ is easily seen to be a basis. □

We next provide an algorithm that works uniformly over all triangular systems. The result is a finite list of pairs of subelliptic multipliers; the length of the list is the multiplicity from (23). The first pair of multipliers is (A_1, B_1) where both A_1 and B_1 equal the Jacobian. The last pair is $(1, 1)$. The number of pairs in the list is exactly the multiplicity (or length) of the ideal (h). The key point is that each A_j is obtained from B_j by taking a controlled root of some of its factors. In other words, each B_j divides a power of A_j, and the power never exceeds the dimension.

We remark that the proof appears at first glance to be inefficient, as delicate machinations within it amount to lowering an exponent by one. This inefficiency arises because the proof works uniformly over all choices of the g_{ij} in (21). Perhaps the proof could be rewritten as an induction on the multiplicity.

Theorem 5.1. *There is an effective algorithm for establishing subelliptic estimates for (domains defined by) triangular systems. That is, let $h_1, \ldots, h_n$ define a triangular system with $L = \mathbf{D}(h) = \prod m_j$. The following hold:*

1) *There is a finite list of pairs of subelliptic multipliers $(B_1, A_1), \ldots, (B_L, A_L)$ such that $B_1 = A_1 = \det(\frac{\partial h_i}{\partial z_j})$, also $B_L = A_L$, and B_L is a unit.*
2) *Each B_j divides a power of A_j. The power depends on only the dimension n and not on the functions h_j. In fact, we never require any power larger than n.*
3) *The length L of the list equals the multiplicity $\mathbf{D}(h)$ given in (23).*

Proof. The proof is a complicated multiple induction. For clarity we write out the cases $n = 1$ and $n = 2$ in full.

When $n = 1$ we never need to take radicals. When $n = 1$ we may assume $h_1(z_1) = z_1^{m_1}$. We set $B_1 = A_1 = (\frac{\partial}{\partial z_1})h_1$, and we set $B_j = A_j = (\frac{\partial}{\partial z_1})^j h_1$. Then B_1 is a subelliptic multiplier, and each B_{j+1} is the derivative of B_j and hence also a subelliptic multiplier; it is the determinant of the one-by-one matrix given by dB_j. Since h_1 vanishes to order m_1 at the origin, the function B_{m_1} is a non-zero constant. Thus 1) holds. Here $L = m_1$ and hence 3) holds. Since $B_j = A_j$ we also verify that the power used never exceeds the dimension, and hence 3) holds. Thus the theorem holds when $n = 1$.

We next write out the proof when $n = 2$. The initial allowable rows are dh_1 and dh_2, giving a lower triangular two-by-two matrix, because $\frac{\partial h_1}{\partial z_2} = 0$. We set

$$B_1 = A_1 = \det(\frac{\partial h_i}{\partial z_j}) = Dh_1 Dh_2,$$

where we use the following convenient notation:

$$Dh_k = \frac{\partial h_k}{\partial z_k}. \tag{24}$$

For $1 \le j \le m_2$ we set

$$B_j = (Dh_1)^2 \ D^j h_2 \tag{25.1}$$

$$A_j = Dh_1 \ D^j h_2. \tag{25.2}$$

Each B_{j+1} is a subelliptic multiplier, obtained by taking the determinant of the allowable matrix whose first row is dh_1 and second row is dA_j. Recall that $D^{m_2}h_2$ is a unit. When $j = m_2$ in (25.2) we therefore find that A_{m_2} is a unit times Dh_1. The collection of multipliers is an ideal, and hence Dh_1 is a subelliptic multiplier. We may use $d(Dh_1)$ as a new allowable first row. Therefore

$$B_{m_2+1} = D^2(h_1)Dh_2.$$

Using $d(h_1)$ as the first row and $d(B_{m_2+1})$ as the second row, we obtain

$$B_{m_2+2} = (D^2h_1)^2 \ D^2h_2$$

$$A_{m_2+2} = D^2h_1 \ D^2h_2.$$

Notice again that we took only a square root of the first factor; more precisely, A_k^2 is divisible by B_k, where $k = m_2 + 2$. Thus each A_k is a multiplier as well. Proceeding in this fashion we obtain

$$A_{m_2+j} = D^2(h_1)D^j h_2,$$

and therefore A_{2m_2} is a unit times $D^2(h_1)$. Thus $d(D^2h_1)$ is an allowable row. We increase the index by m_2 in order to differentiate h_1 once! Applying this procedure a total of m_1 times we see that $B_{m_1m_2}$ is a unit.

We started with $A_1 = B_1$; otherwise each B_j divides A_j^2. Since each B_j is a determinant of a matrix of allowable rows, each B_j is a subelliptic multiplier. Therefore each A_j is a subelliptic multiplier, and $A_L = B_L$ is a unit when $L = m_1m_2$. We have verified 1), 2), and 3).

We pause to repeat why we needed to take radicals of order two in the above. After establishing that $A_j = Dh_1D^jh_2$ is a multiplier, we use dA_j as an allowable row. The next determinant becomes $v = (Dh_1)^2 D^{j+1} h_2$. If we use v as a multiplier, then we obtain both Dh_1 and D^2h_1 as factors. Instead we replace v with $Dh_1 D^{j+1} h_2$ in order to avoid having both Dh_1 and D^2h_1 appear.

We now describe this aspect of the process when $n = 3$ before sketching the induction. For $n = 3$, we will obtain

$$A_1 = B_1 = (Dh_1)(Dh_2)(Dh_3).$$

After m_3 steps we will find that A_{m_3} is a unit times $Dh_1\ Dh_2$. To compute the next determinant we use dh_1 as the first row, $d(Dh_1\ Dh_2)$ as the second row, and dA_1 as the third row. Each of these includes Dh_1 as a factor, and hence $(Dh_1)^3$ is a factor of the determinant. Hence we need to take a radical of order three.

For general n, each matrix of allowable rows used in this process is lower triangular, and hence each determinant taken is a product of precisely n expressions. As above, the largest number of repeated factors is precisely equal to the dimension.

Now we make the induction hypothesis: we assume that $n \geq 2$, and that $h_1, \ldots, h_n$ defines a triangular system. We assume that 1) and 2) hold for all triangular systems in $n-1$ variables. We set

$$B_1 = A_1 = \det(\frac{\partial h_i}{\partial z_j}) = Dh_1 Dh_2 \cdots Dh_n. \tag{26}$$

We replace the last allowable row by dA_n and take determinants, obtaining

$$B_2 = Dh_1 Dh_2 \cdots Dh_{n-1}\ Dh_1 Dh_2 \cdots Dh_{n-1} D^2 h_n \tag{27}$$

as a subelliptic multiplier. Taking a root of order two, we obtain

$$A_2 = Dh_1 Dh_2 \cdots Dh_{n-1} D^2 h_n \tag{28}$$

as a subelliptic multiplier. Repeating this process m_n times we obtain

$$A_{m_n} = Dh_1 Dh_2 \cdots Dh_{n-1} \tag{29}$$

as a subelliptic multiplier. We use its differential dA_{m_n} as the $n-1$-st allowable row, and use dh_n as the nth allowable row. Taking determinants shows that

$$A_{m_n+1} = Dh_1 Dh_2 \cdots Dh_{n-2} Dh_1 Dh_2 \cdots D^2 h_{n-1} Dh_n \tag{30}$$

is a subelliptic multiplier.

What we have done? We are in the same situation as before, but we have differentiated the function h_{n-1} one more time, and hence we have taken one step in decreasing the multiplicity of the singularity. We go through the same process $m_{n-1}m_n$ times and we determine that $A_{m_n m_{n-1}}$ is a subelliptic multiplier which depends upon only the first $n-2$ variables. We then use its differential as the $n-2$-nd allowable row. We obtain, after $m_n m_{n-1} m_{n-2}$ steps, a nonzero subelliptic multiplier independent of the last three variables. By another induction, after $\prod m_j$ steps, we obtain a unit. Each determinant is the product of n diagonal elements. At any stage of the process we can have a fixed derivative of h_1 appearing as a factor to at most the first power in each of the diagonal elements. Similarly a derivative of Dh_2 can occur as a factor only in the last $n-1$ diagonal elements. It follows that we never need to take more than nth root in passing from the B_k (which is a determinant) to the A_k. After L steps in all we obtain the unit

$$D^{m_1}h_1D^{m_2}h_2\dots D^{m_n}h_n = A_L = B_L$$

as a subelliptic multiplier. Thus 1), 2), and 3) hold. □

Corollary 5.1. *Let Ω be a domain defined near 0 by*

$$\mathrm{Re}(z_{n+1}) + \sum |h_j(z)|^2,$$

where h_j are as in (21). There is $\epsilon > 0$ such that the subelliptic estimate (4) holds at 0 for all choices of the arbitrary function g_{jk} in (21).

The algorithm used in the proof of Theorem 5.1 differs from Kohn's algorithm. At each stage we choose a single function A with two properties. Some power of A is divisible by the determinant of a matrix of allowable rows, and the differential dA provides a new allowable row. The algorithm takes exactly $\mathbf{D}(h)$ steps. Thus we do not consider the modules $\mathcal{M}_k$; instead we add one row at a time to the list of allowable $(1,0)$ forms. By being so explicit we avoid the uncontrolled radicals required in Proposition 4.2.

Remark 5.2. The difference in this approach from [K4] can be expressed as follows. We replace the use of uncontrolled radicals by allowing only nth roots of specific multipliers. On the other hand, we must pay by taking derivatives more often. The special case when $n = 1$ clarifies the difference.

The multiplicity $\mathbf{D}(h)$ is the dimension over $\mathbb{C}$ of the quotient algebra $R/(h)$. This algebra plays an important role in commutative algebra, and it is worth noticing that the process in Theorem 5.1 seems to be moving through basis elements for this algebra as it finds the A_j. We note however that the multipliers B_j might be in the ideal and hence 0 in the algebra. We give a simple example.

Example 5.2. Let $h(z,w) = (z^2, w^2)$. The multiplicity is 4. We follow the proof of Theorem 5.1. We have $(A_1, B_1) = (zw, zw)$. We have $(A_2, B_2) = (z, z^2)$. We have $(A_3, B_3) = (w, w^2)$, and finally $(A_4, B_4) = (1, 1)$. Notice that the A_j give the basis for the quotient algebra, whereas two of the B_j lie in the ideal (h).

To close this section we show that we cannot obtain 1 as a subelliptic multiplier when the initial set does not define an $\mathbf{m}$-primary ideal. This result indicates why the presence of complex analytic curves in the boundary precludes subelliptic estimates on $(0,1)$ forms. In Theorem 6.2 we state a more precise result from [C1].

Proposition 5.1. *Let $h_j \in \mathbf{m}$ for each j, and suppose $(h_1, \dots, h_K)$ is not $\mathbf{m}$-primary. Then the stabilized ideal from the algorithm is not the full ring R_n.*

Proof. Since the (analytic or formal) variety defined by the h_j is positive dimensional, we can find a (convergent or formal) nonconstant n-tuple of power series in one variable t, written $z(t)$, such that $h_j(z(t)) = 0$ in R_1 for all j. Differentiating yields

$$\sum \frac{\partial h_j}{\partial z_k}(z(t))z'_k(t) = 0. \tag{31}$$

Hence the matrix $\frac{\partial h_j}{\partial z_k}$ has a nontrivial kernel, and so each of its n by n minor determinants J vanishes after substitution of $z(t)$. Since $J(z(t)) = 0$,

$$\sum \frac{\partial J}{\partial z_k}(z(t))z'_k(t) = 0. \tag{32}$$

Hence including the 1-form dJ does not change the collection of vectors annihilated by a matrix of allowable rows. Continuing we see that $z'(t)$ lies in the kernel of all new matrices we form from allowable rows, and hence $g(z(t))$ vanishes for all functions g in the stabilized ideal. Since $z(t)$ is not constant, we conclude that the variety of the stabilized ideal is positive dimensional, and hence the stabilized ideal is not R_n. □

6. Necessary and sufficient conditions for subellipticity

In the previous sections we have seen a sufficient condition for subellipticity. A subelliptic estimate holds if and only if the function 1 is a subelliptic multiplier; there is an algorithmic procedure to construct subelliptic multipliers beginning with the defining function and the determinant of the Levi form. Each step of the process decreases the value of ϵ known to work in (4). If, however, the process terminates in finitely many steps, then (4) holds for some positive ϵ. Using an important geometric result from [DF], Kohn [K4] established that the process must terminate when the boundary is real-analytic, and that 1 is a subelliptic multiplier if and only if there is no complex variety of positive dimension passing through p and lying in the boundary.

In this section we recall from [C1], [C2], [C3] a different approach to these estimates. The sufficient condition for an estimate involves the existence of plurisubharmonic functions with certain properties. Such functions can be used as weight functions in proving L^2 estimates. See also [He]. A related approach to the estimates appears in [S]; existence of good plurisubharmonic functions implies subellipticity. This intuitive method even works on domains with Lipschitz boundaries.

We wish to relate the estimate (4) to the geometry of the boundary. Let r be a smooth local defining function of a pseudoconvex domain Ω, and assume $0 \in b\Omega$. We consider families $\{M_t\}$ of holomorphic curves through p and how these curves contact $b\Omega$ there. For $t > 0$ we consider nonsingular holomorphic curves g_t as follows:

1) $g_t : \{|\zeta| < t\} \to \mathbb{C}^n$ and $g_t(0) = 0$.
2) There is a positive constant c_2 (independent of t) such that, on $\{|\zeta| < 1\}$, we have $|g'_t(\zeta)| \leq c_2$.
3) There is a positive constant c_1 such that $c_1 \leq |g'_t(0)|$.

We say that the *order of contact* of the family $\{M_t\}$ (of holomorphic curves parametrized by g_t) with $b\Omega$ is η_0 if η_0 is the supremum of the set of real numbers η for which

$$\sup_\zeta |r(g_t(\zeta))| \leq Ct^\eta. \tag{33}$$

The holomorphic curves g_t considered in this definition are all nonsingular. Therefore this approach differs somewhat from the approach in [D1] and [D2], where allowing germs of curves with singularities at 0 is crucial. Our next example provides some insight.

Example 6.1. 1) Define r as follows:

$$r(z) = \mathrm{Re}(z_3) + |z_1^2 - z_2 z_3|^2 + |z_2|^4. \tag{34}$$

By [D1] we have $\mathbf{T}(b\Omega, 0) = 4$. Each curve $\zeta \to g(\zeta)$ whose third component vanishes has contact 4 at the origin. On the other hand, consider a nearby boundary point of the form $(0, 0, ia)$ for a real. Then the curve

$$\zeta \to (\zeta, \frac{\zeta^2}{ia}, ia) = g_a(\zeta) \tag{35}$$

has order of contact 8 at $(0, 0, ia)$. By [D2] this jump is the maximum possible; see (39) below for the sharp inequality in general.

2) Following [C1] we jazz up this example by considering

$$r(z) = \mathrm{Re}(z_3) + |z_1^2 - z_2 z_3^l|^2 + |z_2|^4 + |z_1 z_3^m|^2 \tag{36}$$

for positive integers l, m with $2 \leq l \leq m$. Again we have $\mathbf{T}(b\Omega, 0) = 4$. We will construct a family of regular holomorphic curves g_t with order of contact $\frac{4(2m+l)}{m+2l}$. For $|\zeta| < t$, and α to be chosen, put

$$g_t(\zeta) = (\zeta, \frac{\zeta^2}{(it^\alpha)^l}, it^\alpha). \tag{37}$$

Then, pulling back r to g_t we obtain

$$r(g_t(\zeta)) = \frac{|\zeta|^8}{|t|^{4\alpha l}} + |\zeta|^2 |t|^{2\alpha m}. \tag{38}$$

Setting the two terms in (38) equal, we obtain $|\zeta|^6 = |t|^{4\alpha l + 2\alpha m}$. Put $\alpha = \frac{3}{m+2l}$ and then we get $|\zeta| = |t|$. It follows that

$$\sup_\zeta |r(g_t(\zeta))| = 2|t|^\eta,$$

where $\eta = \frac{4(2m+l)}{m+2l}$. Hence the order of contact of this family is at least η; in fact it is precisely this value. Furthermore, by Theorems 6.1 and 6.2 below, there is a subelliptic estimate at 0 for $\epsilon = \frac{1}{\eta}$ and this value is the largest possible. Depending on l and m, the possible values of the upper bound on ϵ live in the interval $[\frac{1}{8}, \frac{1}{4}]$.

We next recall the equivalence of subelliptic estimates on $(0,1)$ forms with finite type.

Theorem 6.1. *See* [C2] *and* [C3]. *Suppose that $b\Omega$ is smooth and pseudoconvex and $\mathbf{T}(b\Omega, p_0)$ is finite. Then the subelliptic estimate* (4) *holds for some $\epsilon > 0$.*

Theorem 6.2. *See* [C1]. *Suppose that the subelliptic estimate* (4) *holds for some positive ϵ. If $\{M_t\}$ is a family of complex-analytic curves of diameter t, then the order of contact of $\{M_t\}$ with $b\Omega$ is at most $\frac{1}{\epsilon}$.*

In two dimensions, type of a point is an upper semi-continuous function: if the type at p is t, then the type is at most t nearby. In higher dimensions the type of a nearby point can be larger (as well as the same or smaller). Sharp local bounds for the type indicate why relating the supremum of possible values of ϵ in a subelliptic estimate to the type is difficult in dimension at least 3.

Theorem 6.3. *See* [D2]. *Let $b\Omega$ in $\mathbb{C}^n$ be smooth and pseudoconvex near p_0, and assume $\mathbf{T}(b\Omega, p_0)$ is finite. Then there is a neighborhood of p_0 on which*

$$\mathbf{T}(b\Omega, p) \leq \frac{\mathbf{T}(b\Omega, p_0)^{n-1}}{2^{n-2}}. \tag{39}$$

The bound (39) is sharp. When $n = 2$ we see that the type at a nearby point can be no larger than the type at p_0. When $n \geq 3$, however, the type can be larger nearby. This failure of upper semi-continuity of the type shows that the best epsilon in a subelliptic estimate cannot simply be the reciprocal of the type, as holds in two dimensions. See [D1], [D2], [D3] for more information. Example 7.1 below generalizes Example 6.1. It is an unpublished result due to the first author.

All these examples are based upon a simple example found by the second author in [D3] to illustrate the failure of upper semi-continuity of order of contact. See [D1] and [D2] for extensions to higher dimensions and a proof of (39).

7. Sharp subelliptic estimates

Example 7.1. Consider the local defining function r given by

$$r(z) = 2\mathrm{Re}(z_3) + |z_1^{m_1} - f(z_3)z_2|^2 + |z_2^{m_2}|^2 + |z_2 g(z_3)|^2, \tag{40}$$

where m_1 and m_2 are integers at least 2 and f and g are functions to be chosen. Let $b\Omega$ be the zero set of r. Assume $f(0) = g(0) = 0$. It follows by [D1] that $\mathbf{T}(b\Omega, 0) = 2\max(m_1, m_2)$ and $\mathrm{mult}(b\Omega, 0) = 2m_1m_2$. We will show that we can obtain, for the reciprocal of the largest possible value of ϵ in a subelliptic estimate, any value in between these two numbers.

By Theorem 6.1 there is a subelliptic estimate. According to Theorem 6.2, to find an upper bound for ϵ we must find a family $\{M_t\}$ of one-dimensional complex curves with certain properties. We follow Example 6.1 and define this family $\{M_t\}$

as follows: M_t is the image of the holomorphic curve

$$\gamma_t(\zeta) = (\zeta, \frac{\zeta^{m_1}}{f(it)}, it) \tag{41}$$

on the set where $|\zeta| \leq t$.

Pulling back r to this family of curves yields

$$r(\gamma_t(\zeta)) = |\frac{\zeta^{m_1}}{f(it)}|^{2m_2} + |\frac{\zeta^{m_1}}{f(it)}|^2 |g(it)|^2. \tag{42}$$

Reasoning as in Example 6.1, we choose f and g to make the two terms in (42) equal. The condition for equality is

$$|\zeta|^{2m_1m_2-2m_1} = |f|^{2m_2-2}|g|^2. \tag{43}$$

The crucial difference now is that the functions f and g, which depend on only one variable, can be chosen as we wish. In particular, choose a parameter $\lambda \in (0, 1]$, and assume that f and g are chosen such that $\log(|f|) = \lambda\log(|g|)$. Then (43) gives

$$(2m_1(m_2 - 1))\log(|\zeta|) = (2(m_2 - 1)\lambda + 2)\log(|g|). \tag{44}$$

We obtain from (44) and (45)

$$\log(\frac{|r(\gamma_t(\zeta))|}{2}) = \left(2m_1 + \frac{(2 - 2\lambda)2m_1(m_2 - 1)}{2(m_2 - 1)\lambda + 2)}\right)\log(|\zeta|). \tag{45}$$

In order to find a value η for which there is a constant C such that $|r| \leq C|\zeta|^\eta$, we take logs and see that we find the ratio $\frac{\log(|r|)}{\log(|\zeta|)}$. Using (45) we obtain the order of contact $\mathbf{T}$ of this family of curves to be

$$\mathbf{T} = 2m_1 + \frac{2(1 - \lambda)m_1(m_2 - 1)}{(m_2 - 1)\lambda + 1}. \tag{46}$$

If in (46) we put $\lambda = 1$ then we get $\mathbf{T} = 2m_1$. If in (46) we let λ tend to 0, we obtain $\mathbf{T} = 2m_1 + 2m_1(m_2 - 1) = 2m_1m_2$.

In the previous example we may, for example, choose $f(z) = z^p$ and $g(z) = z^q$. If we put $\lambda = \frac{p}{q}$, then our calculations apply, and (46) is rational. On the other hand, we can achieve the condition $\log(|f|) = \lambda\log(|g|)$ by allowing f and g to be functions vanishing to infinite order at 0 but which are holomorphic in the half-plane $\mathrm{Re}(z_3) < 0$. For example we may define f by $f(\zeta) = \exp(\frac{-p}{\sqrt{-\zeta}})$ and g the same except that p is replaced by q. By doing so we can allow λ in (46) to be real. It is easy to include the limiting value $\lambda = 0$, by setting $g = 0$.

In order to finish we have to discuss sufficiency. The first author uses the method of weighted L^2 estimates. We let $H(\Phi)$ denote the complex Hessian of a smooth real-valued function Φ. We say that $H(\Phi) \geq C$ if the minimum eigenvalue of the Hessian is at least C at each point. One of the crucial steps in the proof of Theorem 6.1 is the following result from [C2], based upon ideas from [C4].

Theorem 7.1. *Let Ω be a smoothly bounded domain, defined near a boundary point p by $\{r = 0\}$. Suppose that there is neighborhood U of p such that the following holds: For each $\delta > 0$, we can find a smooth function Φ_δ satisfying*

1) $|\Phi_\delta| \leq 1$ *on U. Thus Φ_δ is uniformly bounded.*
2) Φ_δ *is plurisubharmonic on U. Thus $H(\Phi_\delta) \geq 0$ on U.*
3) $H(\Phi_\delta) \geq c\delta^{-2\epsilon}$ *on $U \cap \{-\delta < r \leq 0\}$. Thus the Hessian of Φ_δ blows up in a precise manner as we approach the boundary.*

Then there is a subelliptic estimate of order ϵ at p.

Using this result it is possible to say more about Example 7.1. One can choose f and g there such that there is a subelliptic estimate of order ϵ at the origin, where ϵ is the reciprocal of the number $\mathbf{T}$ in (46). In particular, for every ϵ_0 in the range $[\frac{1}{2m_1m2}, \frac{1}{2m_1}]$ there is a domain in $\mathbb{C}^3$ such that the largest possible value of ϵ in a subelliptic estimate is ϵ_0. By changing the function g appropriately, one can create the situation of part 2) of the next result.

Theorem 7.2. *Let ϵ_0 be in the interval $(0, \frac{1}{4}]$.*

1) *There is a smooth pseudoconvex domain in $\mathbb{C}^3$, with defining function* (40)*, such that the subelliptic estimate* (4) *holds with ϵ equal to ϵ_0, but for no larger value of ϵ. In addition, if ϵ_0 (in the same range) is rational, then we can choose the domain to be defined by* (40)*, where $f(z) = z^p$ and $g(z) = z^q$, and hence the defining equation is a polynomial.*

2) *There is also a smooth pseudoconvex domain in $\mathbb{C}^3$, with defining equation* (40)*, such that the estimate* (4) *holds for all ϵ with $0 \leq \epsilon < \epsilon_0$, but for which the estimate fails at ϵ_0.*

Theorem 7.2 can be extended to higher dimensions. It is much harder to understand subelliptic estimates on $(0, 1)$ forms in three or more dimensions than it is in two dimensions. The theory for $(0, 1)$ forms in two dimensions is analogous to the theory for $(0, n-1)$ forms in n dimensions. In these cases there is no need to consider the contact with singular varieties, and hence issues involving subelliptic estimates are controlled by commutators. We conclude by observing that connections between the analysis and the commutative algebra involved do not reveal themselves in two dimensions, or more generally, when we consider estimates on $(0, n-1)$ forms. Hence Theorem 3.1 tells only a small part of the full story.

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Invariant CR Mappings

John P. D'Angelo

Dedicated to Linda Rothschild

Abstract. This paper concerns CR mappings between hyperquadrics. We emphasize the case when the domain is a sphere and the maps are invariant under a finite subgroup of the unitary group. We survey known results, provide some new examples, and use them to prove the following result. For each d there is a CR polynomial mapping of degree $2d+1$ between hyperquadrics that preserves the number of negative eigenvalues. Thus rigidity fails for CR mappings between hyperquadrics.

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1. Introduction

The subject of CR Geometry interacts with nearly all of mathematics. See [BER] for an extensive discussion of many aspects of CR manifolds and mappings between them. One aspect of the subject not covered in [BER] concerns CR mappings invariant under groups. The purpose of this paper is to discuss interactions with number theory and combinatorics that arise from the seemingly simple setting of group-invariant CR mappings from the unit sphere to a hyperquadric. Some elementary representation theory also arises.

The unit sphere S^{2n-1} in complex Euclidean space $\mathbf{C}^n$ is the basic example of a CR manifold of hypersurface type. More generally we consider the hyperquadric $Q(a,b)$ defined to be the subset of $\mathbf{C}^{a+b}$ defined by

$$\sum_{j=1}^{a} |z_j|^2 - \sum_{j=a+1}^{a+b} |z_j|^2 = 1. \tag{1}$$

Of course S^{2n-1} is invariant under the unitary group $U(n)$. Let Γ be a finite subgroup of $U(n)$. Assume that $f : \mathbf{C}^n \to \mathbf{C}^N$ is a rational mapping invariant under Γ and that $f(S^{2n-1}) \subset S^{2N-1}$. For most Γ such an f must be a constant. In other words, for most Γ, there is no non-constant Γ-invariant rational mapping from sphere to sphere for any target dimension. In fact, for such a non-constant invariant map to exist, Γ must be cyclic and represented in a rather restricted fashion. See [Li1], [DL], and especially Corollary 7 on page 187 of [D1], for precise statements and the considerable details required.

The restriction to rational mappings is natural; for $n \geq 2$ Forstneric [F1] proved that a proper mapping between balls, with sufficiently many continuous derivatives at the boundary, must be a rational mapping. On the other hand, if one makes no regularity assumption at all on the map, then (see [Li2]) one can create group-invariant proper mappings between balls for any fixed-point free finite unitary group. The restrictions on the group arise from CR Geometry and the smoothness of the CR mappings considered. In this paper we naturally restrict our considerations to the class of rational mappings. See [F2] for considerable discussion about proper holomorphic mappings and CR Geometry.

In order to find group-invariant CR mappings from a sphere, we relax the assumption that the target manifold be a sphere, and instead allow it to be a hyperquadric. We can then always find polynomial examples, as we note in Corollary 1.1. In this paper we give many examples of invariant mappings from spheres to hyperquadrics. Our techniques allow us to give some explicit surprising examples. In Theorem 6.1 for example, we show that *rigidity* fails for mappings between hyperquadrics; we find non-linear polynomial mappings between hyperquadrics with the same number of negative eigenvalues in the defining equations of the domain and target hyperquadrics. As in the well-known case of maps between spheres, we must allow sufficiently many positive eigenvalues in the target for such maps to exist.

To get started we recall that a polynomial $R : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}$ is called Hermitian symmetric if $R(z, \overline{w}) = \overline{R(w, \overline{z})}$ for all z and w. If R is Hermitian symmetric, then $R(z, \overline{z})$ is evidently real valued. By polarization, the converse also holds. We note also that a polynomial in $z = (z_1, \ldots, z_n)$ and $\overline{z}$ is Hermitian symmetric if and only if its matrix of coefficients is Hermitian symmetric in the sense of linear algebra.

The following result from [D1] shows how to construct group-invariant mappings from spheres to hyperquadrics. Throughout the paper we will give explicit formulas in many cases.

Theorem 1.1. *Let Γ be a finite subgroup of $U(n)$ of order p. Then there is a unique Hermitian symmetric Γ-invariant polynomial $\Phi_\Gamma(z, \overline{w})$ such that the following hold:*

1) $\Phi_\Gamma(0, 0) = 0$.
2) *The degree of Φ_Γ in z is p.*
3) $\Phi_\Gamma(z, \overline{z}) = 1$ *when z is on the unit sphere.*
4) $\Phi_\Gamma(\gamma z, \overline{w}) = \Phi_\Gamma(z, \overline{w})$ *for all $\gamma \in \Gamma$.*

Corollary 1.1. *There are holomorphic vector-valued* Γ*-invariant polynomial mappings* F *and* G *such that we can write*

$$\Phi_\Gamma(z, \overline{z}) = ||F(z)||^2 - ||G(z)||^2. \tag{2}$$

The polynomial mapping $z \to (F(z), G(z))$ *restricts to a* Γ*-invariant mapping from* S^{2n-1} *to the hyperquadric* $Q(N_+, N_-)$*, where these integers are the numbers of positive and negative eigenvalues of the matrix of coefficients of* Φ_Γ*.*

The results in this paper revolve around how the mapping (F, G) from Corollary 1.1 depends on Γ. We clarify one point at the start; even if we restrict our considerations to cyclic groups, then this mapping changes (surprisingly much) as the representation of the group changes. The interesting things from the points of view of CR Geometry, Number Theory, and Combinatorics all depend in non-trivial ways on the particular representation. Therefore the results should be considered as statements about the particular subgroup $\Gamma \subset U(n)$, rather than as statements about the abstract group G for which $\pi(G) = \Gamma$.

The proof of Theorem 1 leads to the following formula for Φ_Γ.

$$\Phi_\Gamma(z, \overline{w}) = 1 - \prod_{\gamma \in \Gamma} (1 - \langle \gamma z, w \rangle). \tag{3}$$

The first three properties from Theorem 1 are evident from (3), and the fourth property is not hard to check. One also needs to verify uniqueness.

The starting point for this paper will therefore be formula (3). We will first consider three different representations of cyclic groups and we note the considerable differences in the corresponding invariant polynomials. We also consider metacyclic groups. We also discuss some interesting asymptotic considerations, as the order of the group tends to infinity. Additional asymptotic results are expected in appear in the doctoral thesis [G] of D. Grundmeier.

An interesting result in this paper is the application in Section 6. In Theorem 6.1 we construct, for each odd $2p+1$ with $p \geq 1$, a polynomial mapping g_p of degree $2p$ such that

$$g_p : Q(2, 2p+1) \to Q(N(p), 2p+1). \tag{4}$$

These mappings illustrate a failure of *rigidity*; in many contexts restrictions on the eigenvalues of the domain and defining hyperquadrics force maps to be linear. See [BH]. Our new examples show that rigidity does not hold when we keep the number of negative eigenvalues the same, as long as we allow a sufficient increase in the number of positive eigenvalues. On the other hand, by a result in [BH], the additional restriction that the mapping preserves sides of the hyperquadric does then guarantee rigidity. It is quite striking that the construction of the polynomials in Theorem 6.1 relies on the group-theoretic methods in the rest of the paper.

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2. Properties of the invariant polynomials

Let Γ be a finite subgroup of the unitary group $U(n)$, and let Φ_Γ be the unique polynomial defined by (3). Our primary interest concerns how this polynomial depends on the particular representation of the group.

We remark at the outset that we will be considering *reducible* representations. A simple example clarifies why. If G is cyclic of order p, then G has the irreducible unitary (one-dimensional) representation Γ as the group of pth roots of unity. An elementary calculation shows that the invariant polynomial Φ_Γ becomes simply

$$\Phi_\Gamma(z, \overline{w}) = (z\overline{w})^p. \tag{5}$$

On the other hand, there are many ways to represent G as a subgroup of $U(n)$ for $n \geq 2$. We will consider these below; for now we mention one beautiful special case.

Let p and q be positive integers with $1 \leq q \leq p-1$ and let ω be a primitive pth root of unity. Let $\Gamma(p,q)$ be the cyclic group generated by the diagonal 2-by-2 matrix A with eigenvalues ω and ω^q:

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}. \tag{6}$$

Because A is diagonal, the invariant polynomial $\Phi_{\Gamma(p,q)}(z, \overline{z})$ depends on only $|z_1|^2$ and $|z_2|^2$. If we write $x = |z_1|^2$ and $y = |z_2|^2$, then we obtain a corresponding polynomial $f_{p,q}$ in x and y. This polynomial has integer coefficients; a combinatorial interpretation of these coefficients appears in [LWW]. The crucial idea in [LWW] is the interpretation of Φ_Γ as a circulant determinant; hence permutations arise and careful study of their cycle structure leads to the combinatorial result. Asymptotic information about these integers as p tends to infinity appears in both [LWW] and [D4]; the technique in [D4] gives an analogue of the Szegö limit theorem. In the special case where $q = 2$, these polynomials provide examples of sharp degree estimates for proper monomial mappings between balls. The polynomials $f_{p,2}$ have many additional beautiful properties. We pause to write down the formula and state an appealing corollary. These polynomials will arise in the proof of Theorem 6.1.

$$f_{p,2}(x,y) = (-1)^{p+1}y^p + \left(\frac{x+\sqrt{x^2+4y}}{2}\right)^p + \left(\frac{x-\sqrt{x^2+4y}}{2}\right)^p. \tag{7}$$

Corollary 2.1 (D4). *Let S_p be the sum of the coefficients of $f_{p,2}$. Then the limit as p tends to infinity of $S_p^{\frac{1}{p}}$ equals the golden ratio $\frac{1+\sqrt{5}}{2}$.*

Proof. The sum of the coefficients is $f_{p,2}(1,1)$, so put $x = y = 1$ in (7). The largest (in absolute value) of the three terms is the middle term. Taking pth roots and letting p tend to infinity gives the result. □

See [DKR] for degree estimates and [DLe] for number-theoretic information concerning uniqueness results for degree estimates. The following elegant primality test was proved in [D2].

Theorem 2.1. *For each q, the congruence $f_{p,q}(x,y) \cong x^p + y^p \bmod (p)$ holds if and only if p is prime.*

We make a few comments. When $q = 1$, the polynomial $f_{p,1}$ is simply $(x+y)^p$ and the result is well known. For other values of q the polynomials are more complicated. When $q = 2$ or when $q = p - 1$ there are explicit formulas for the integer coefficients. For small q recurrences exist but the order of the recurrences grows exponentially with q. See [D2], [D3], [D4] and [G]. There is no known general formula for the integer coefficients. Nonetheless the basic theory enables us to reduce the congruence question to the special case. Note also that the quotient space $L(p,q) = S^3/\Gamma$ is a Lens space. It might be interesting to relate the polynomials $f_{p,q}$ to the differential topology of these spaces.

We return to the general situation and repeat the crucial point; the invariant polynomials depend on the representation in non-trivial and interesting ways, even in the cyclic case. In order to express them we recall ideas that go back to E. Noether. See [S] for considerable discussion. Given a subgroup Γ of the general linear group, Noether proved that the algebra of polynomials invariant under Γ is generated by polynomials of degree at most the order $|\Gamma|$ of Γ. Given a polynomial p we can create an invariant polynomial by averaging p over the group:

$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} p\circ\gamma. \tag{8}$$

We find a basis for the algebra of invariant polynomials as follows. We average each monomial z^α of total degree at most $|\Gamma|$ as in (8) to obtain an invariant polynomial; often the result will be the zero polynomial. The nonzero polynomials that result generate the algebra of polynomials invariant under Γ. In particular, the number of polynomials required is bounded above by the dimension of the space of homogeneous polynomials of degree $|\Gamma|$ in n variables. Finally we can express the F and G from (2) in terms of sums and products of these basis elements. The invariant polynomials here are closely related to the Chern orbit polynomials from [S]. The possibility of polarization makes our approach a bit different. It seems a worthwhile project to deepen this connection. Some results in this direction will appear in [G].

3. Cyclic groups

Let Γ be cyclic of order p. Then the elements of Γ are $I, A, A^2, \ldots, A^{p-1}$ for some unitary matrix A. Formula (3) becomes

$$\Phi_\Gamma(z,\overline{w}) = 1 - \prod_{j=0}^{p-1}(1 - \langle A^j z, w\rangle). \tag{9}$$

We begin by considering three different representations of a cyclic group of order six; we give precise formulas for the corresponding invariant polynomials.

Let ω be a primitive sixth-root of unity, and let η be a primitive third-root of unity. We consider three different unitary matrices; each generates a cyclic group of order six.

Example 3.1. Let Γ be the cyclic group of order 6 generated by A, where

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}. \tag{10.1}$$

The invariant polynomial satisfies the following formula:

$$\Phi_\Gamma(z, \overline{z}) = (|z_1|^2 + |z_2|^2)^6. \tag{10.2}$$

It follows that Φ is the squared norm of the following holomorphic polynomial:

$$f(z) = (z_1^6, \sqrt{6}z_1^5 z_2, \sqrt{15}z_1^4 z_2^2, \sqrt{20}z_1^3 z_2^3, \sqrt{15}z_1^2 z_2^4, \sqrt{6}z_1 z_2^5, z_2^6). \tag{10.3}$$

The polynomial f restricts to the sphere to define an invariant CR mapping from S^3 to $S^{13} \subset \mathbf{C}^7$.

Example 3.2. Let Γ be the cyclic group of order 6 generated by A, where

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{pmatrix}. \tag{11.1}$$

The invariant polynomial satisfies the following formula:

$$\Phi_\Gamma(z, \overline{z}) = |z_1|^{12} + |z_2|^{12} + 6|z_1|^2|z_2|^2 + 2|z_1|^6|z_2|^6 - 9|z_1|^4|z_2|^4. \tag{11.2}$$

Note that Φ is not a squared norm. Nonetheless we define f as follows:

$$f(z) = (z_1^6, z_2^6, \sqrt{6}z_1 z_2, \sqrt{2}z_1^3 z_2^3, 3z_1^2 z_2^2). \tag{11.3}$$

Then

$$\Phi = |f_1|^2 + |f_2|^2 + |f_3|^2 + |f_4|^2 - |f_5|^2,$$

and the polynomial f restricts to the sphere to define an invariant CR mapping from S^3 to $Q(4, 1) \subset \mathbf{C}^5$.

Example 3.3. Let Γ be the cyclic group of order 6 generated by A, where

$$A = \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix}. \tag{12.1}$$

The polynomial Φ can be expressed as follows:

$$\Phi(z, \overline{z}) = (x + y)^3 + (\eta s + \overline{\eta} t)^3 - (x + y)^3(\eta s + \overline{\eta} t)^3 \tag{12.2}$$

where

$$\begin{aligned} x &= |z_1|^2 \\ y &= |z_2|^2 \\ s &= z_2\overline{z}_1 \\ t &= z_1\overline{z}_2. \end{aligned}$$

After diagonalization this information determines a (holomorphic) polynomial mapping (F, G) such that

$$\Phi_\Gamma = ||F||^2 - ||G||^2.$$

It is somewhat complicated to determine the components of F and G. It is natural to use Noether's approach. For this particular representation, considerable computation then yields the following invariant polynomials:

$$\begin{aligned} z_1^3 + z_2^3 &= p \\ z_1^2 z_2 + \eta z_1 z_2^2 &= q \\ z_1^6 + z_2^6 &= f \\ z_1^3 z_2^3 &= \frac{1}{2}(p^2 - f) \\ z_1 z_2^5 + \eta^2 z_1^5 z_2 &= k \\ \frac{1}{2}(z_1^4 z_2^2 + \eta^2 z_1^2 z_2^4) &= h \end{aligned}$$

In order to write Φ_Γ nicely, we let

$$g = c(z_1 z_2^5 + 3\eta z_1^3 z_2^3 + \eta^2 z_1^5 z_2).$$

Then one can write Φ_Γ, for some $C > 0$ as follows:

$$\phi_\Gamma = |p|^2 + |q|^2 + \frac{1}{2}(|f - z_1^3 z_2^3|^2 - |f + z_1^3 z_2^3|^2) + C(|g - h|^2 - |g + h|^2). \quad (12.3)$$

We conclude that the invariant polynomial determines an invariant CR mapping (F, G) from the unit sphere S^3 to the hyperquadric $Q(4, 2) \subset \mathbf{C}^6$. We have

$$F = \left(p, q, \frac{1}{\sqrt{2}}(f - z_1^3 z_2^3), \sqrt{C}(g - h)\right) \quad (12.4)$$

$$G = \left(\frac{1}{\sqrt{2}}(f + z_1^3 z_2^3), \sqrt{C}(g + h)\right). \quad (12.5)$$

Consider these three examples together. In each case we have a cyclic group of order six, represented as a subgroup of $U(2)$. In each case we found an invariant CR mapping. The image hyperquadrics were $Q(7, 0)$, $Q(4, 1)$, and $Q(4, 2)$. The corresponding invariant mappings had little in common. In the first case, the map was homogeneous; in the second case the map was not homogeneous, although it was a monomial mapping. In the third case we obtained a rather complicated non-monomial map. It should be evident from these examples that the mappings depend in non-trivial ways on the representation.

4. Asymptotic information

In this section we consider three families of cyclic groups, $\Gamma(p, 1)$, $\Gamma(p, 2)$, and $\Gamma(p, p - 1)$. For these groups it is possible to compute the invariant polynomials Φ_Γ exactly. In each case, because the group is generated by a diagonal matrix, the invariant polynomial depends on only $x = |z_1|^2$ and $y = |z_2|^2$. We will therefore often write the polynomials as functions of x and y.

For $p = 1$ we have

$$\Phi_\Gamma(z, \overline{z}) = (|z_1|^2 + |z_2|^2)^p = (x + y)^p. \tag{13}$$

It follows that there is an invariant CR mapping to a sphere, namely the hyperquadric $Q(p + 1, 0)$. We pause to prove (13) by establishing the corresponding general result in any domain dimension.

Theorem 4.1. *Let Γ be the cyclic group generated by ωI, where I is the identity operator on $\mathbf{C}^n$ and ω is a primitive pth root of unity. Then $\Phi_\Gamma(z, \overline{z}) = ||z||^{2p} = ||z^{\otimes p}||^2$. Thus $\Phi_{\Gamma(p,1)}^{\frac{1}{p}} = ||z||^2$ and hence it is independent of p.*

Proof. A basis for the invariant polynomials is given by the homogeneous monomials of degree p. By Theorem 1.1 Φ_Γ is of degree p in z and hence of degree $2p$ overall. It must then be homogeneous of total degree $2p$ and it must take the value 1 on the unit sphere; it therefore equals $||z||^{2p}$. □

We return to the case where $n = 2$ where $||z||^2 = |z_1|^2 + |z_2|^2 = x + y$. In the more complicated situation arising from $\Gamma(p, q)$, the expression $\Phi_{\Gamma(p,q)}^{\frac{1}{p}}$ is not constant, but its behavior as p tends to infinity is completely analyzed in [D4].

As an illustration we perform this calculation when $q = 2$. By expanding (3) the following formula holds (see [D4] for details and precise formulas for the n_j):

$$f_{p,2}(x, y) = \Phi_{\Gamma(p,2)}(z, \overline{z}) = x^p + (-1)^{p+1} y^p + \sum_j n_j x^{p-2j} y^j. \tag{14.1}$$

Here the n_j are positive integers and the summation index j satisfies $2j \leq p$. The target hyperquadric now depends on whether p is even or odd. When $p = 2r - 1$ is odd, the target hyperquadric is the sphere, namely the hyperquadric $Q(r + 1, 0)$. When $p = 2r$ is even, the target hyperquadric is $Q(r + 1, 1)$. In any case, using (7) under the condition $x + \sqrt{x^2 + 4y} > 2y$, we obtain

$$(f_{p,2}(x, y))^{\frac{1}{p}} = \frac{x + \sqrt{x^2 + 4y}}{2} (1 + h_p(x, y))^{\frac{1}{p}}, \tag{14.2}$$

where $h_p(x, y)$ tends to zero as p tends to infinity. Note that we recover Corollary 2.1 by setting $x = y = 1$. We summarize this example in the following result. Similar results hold for the $f_{p,q}$ for $q \geq 3$. See [D4].

Proposition 4.1. *For $x + \sqrt{x^2 + 4y} > 2y$, the limit, as p tends to infinity, of the left-hand side of* (14.2) *exists and equals* $\frac{x+\sqrt{x^2+4y}}{2}$.

It is also possible to compute $\Phi_{\Gamma_{(p,p-1)}}$ exactly. After some computation we obtain the following:

$$\Phi_\Gamma(z, \overline{z}) = |z_1|^{2p} + |z_2|^{2p} + \sum_j n_j (|z_1|^2 |z_2|^2)^j = x^p + y^p + \sum n_j (xy)^j, \tag{15}$$

where the n_j are integers. They are 0 when $2j > p$, and otherwise non-zero. In this range $n_j > 0$ when j is odd, and $n_j < 0$ when j is even. Explicit formulas for the n_j exist; in fact they are closely related to the coefficients for $f_{p,2}$. See [D3]. To see what is going on, we must consider the four possibilities for p modulo (4).

We illustrate by listing the polynomials of degrees $4, 5, 6, 7$. As above we put $|z_1|^2 = x$ and $|z_2|^2 = y$. We obtain:

$$f_{4,3}(x,y) = x^4 + y^4 + 4xy - 2x^2y^2 \tag{16.4}$$

$$f_{5,4}(x,y) = x^5 + y^5 + 5xy - 5x^2y^2 \tag{16.5}$$

$$f_{6,5}(x,y) = x^6 + y^6 + 6xy - 9x^2y^2 + 2x^3y^3 \tag{16.6}$$

$$f_{7,6}(x,y) = x^7 + y^7 + 7xy - 14x^2y^2 + 7x^3y^3. \tag{16.7}$$

For $\Gamma(p, p-1)$ one can show the following. When $p = 4k$ or $p = 4k+1$, we have $k+2$ positive coefficients and k negative coefficients. When $p = 4k+2$ or $p = 4k+3$, we have $k+3$ positive coefficients and k negative coefficients. For $q > 2$ in general one obtains some negative coefficients when expanding $f_{p,q}$, and hence the target must be a (non-spherical) hyperquadric. The paper [LWW] provides a method for determining the sign of the coefficients.

Given a finite subgroup Γ of $U(n)$, the invariant polynomial Φ_Γ is Hermitian symmetric, and hence its underlying matrix of coefficients is Hermitian. We let $N_+(\Gamma)$ denote the number of positive eigenvalues of this matrix, and we let $N_-(\Gamma)$ denote the number of negative eigenvalues. When Γ is cyclic of order p we sometimes write $N_+(p)$ instead of $N_+(\Gamma)$, but the reader should be warned that the numbers N_+ and N_- depend upon Γ and not just p. The ratio $R_p = \frac{N_+(p)}{N_+(p)+N_-(p)}$ is of some interest, but it can be hard to compute. We therefore consider its asymptotic behavior.

For the class of groups considered above Theorem 4.1 holds. It is a special case of a result to appear in the doctoral thesis [G] of Grundmeier, who has found the limit of R_p for many classes of groups (not necessarily cyclic) whose order depends on p. Many different limiting values can occur. Here we state only the following simple version which applies to the three classes under consideration.

Proposition 4.2. *For the three classes of cyclic groups whose invariant polynomials are given by* (13), (14), *and* (15), *the limit of R_p as p tends to infinity exists. In the first two cases the limit is* 1. *When Φ_Γ satisfies* (15), *the limit is* $\frac{1}{2}$.

Remark 4.1. For the class of groups $\Gamma(p, q)$ the limit L_q of R_p exists and depends on q. If one then lets q tend to infinity, the resulting limit equals $\frac{3}{4}$. Thus the asymptotic result differs from the limit obtained by setting $q = p-1$ at the start. The subtlety of the situation is evident.

5. Metacyclic groups

Let C_p denote a cyclic group of order p. A group G is called *metacyclic* if there is an exact sequence of the form

$$1 \to C_p \to G \to C_q \to 1.$$

Such groups are also described in terms of two generators A and B such that $A^p = I$, $B^q = I$, and $AB = B^m A$ for some m. In this section we will consider metacyclic subgroups of $U(2)$ defined as follows. Let ω be a primitive pth root of unity, and let A be the following element of $U(2)$:

$$\begin{pmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{pmatrix}. \tag{17}$$

For these metacyclic groups we obtain in (23) a formula for the invariant polynomials in terms of known invariant polynomials for cyclic groups. We write $C(p, p-1)$ for the cyclic subgroup of $U(2)$ generated by A. Its invariant polynomial is

$$\Phi_{C(p,p-1)} = 1 - \prod_{k=0}^{p} (1 - \langle A^k z, z \rangle). \tag{18}$$

Now return to the metacyclic group Γ. Each group element of Γ will be of the form $B^j A^k$ for appropriate exponents j, k. Since B is unitary, $B^* = B^{-1}$. We may therefore write

$$\langle B^j A^k z, w \rangle = \langle A^k z, B^{-j} w \rangle. \tag{19}$$

We use (19) in the product defining Φ_Γ to obtain the following formula:

$$\Phi_\Gamma(z, \overline{z}) = 1 - \prod_{k=0}^{p-1} \prod_{j=0}^{q-1} (1 - \langle B^j A^k z, z \rangle) = 1 - \prod_{k=0}^{p-1} \prod_{j=0}^{q-1} (1 - \langle A^k z, B^{-j} z \rangle). \tag{20}$$

Notice that the term

$$\prod_{k=0}^{p-1} (1 - \langle A^k z, B^{-j} z \rangle) \tag{21}$$

can be expressed in terms of the invariant polynomial for the cyclic group $C(p, p-1)$. We have

$$\prod_{k=0}^{p-1} (1 - \langle A^k z, B^{-j} z \rangle) = 1 - \Phi_{C(p,p-1)}(z, B^{-j} z), \tag{22}$$

and hence we obtain

$$\Phi_\Gamma(z, \overline{z}) = 1 - \prod_{j=0}^{q-1} \left(1 - \Phi_{C(p,p-1)}(z, B^{-j} z)\right). \tag{23}$$

The invariance of Φ_Γ follows from the definition, but this property is not immediately evident from this polarized formula. The other properties from Theorem 1.1 are evident in this version of the formula. We have $\Phi_\Gamma(0, 0) = 0$. Also, $\Phi_\Gamma(z, \overline{z}) = 1$

on the unit sphere, because of the term when $j = 0$. The degree in z is pq because we have a product of q terms each of degree p.

The simplest examples of metacyclic groups are the dihedral groups. The dihedral group D_p is the group of symmetries of a regular polygon of p sides. The group D_p has order $2p$; it is generated by two elements A and B, which satisfy the relations $A^p = I$, $B^2 = I$, and $AB = BA^{p-1}$. Thus A corresponds to a rotation and B corresponds to a reflection. We may represent D_p as a subgroup of $U(2)$ by putting

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \tag{24.1}$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{24.2}$$

Formula (23) for the invariant polynomial simplifies because the product in (23) has only two terms. We obtain the following result, proved earlier in [D2].

Theorem 5.1. *The invariant polynomial for the above representation of D_p satisfies the following formula:*

$$\Phi(z, \overline{z}) = f_{p,p-1}(|z_1|^2, |z_2|^2) + \\ + f_{p,p-1}(z_2\overline{z}_1, z_1\overline{z}_2) - f_{p,p-1}(|z_1|^2, |z_2|^2) f_{p,p-1}(z_2\overline{z}_1, z_1\overline{z}_2). \tag{25}$$

6. An application; failure of rigidity

In this section we use the group invariant approach to construct the first examples of polynomial mappings of degree $2p$ from $Q(2, 2p+1)$ to $Q(N(p), 2p+1)$. The key point of these examples is that the number of negative eigenvalues is preserved. The mappings illustrate the failure of rigidity in the case where we keep the number of negative eigenvalues the same but we are allowed to increase the number of positive eigenvalues sufficiently. The mappings arise from part of a general theory being developed [DLe2] by the author and J. Lebl. As mentioned in the introduction, the additional assumption that the mapping preserves sides of the hyperquadric does force linearity in this context. [BH]

Theorem 6.1. *Let $2p + 1$ be an odd number with $p \geq 1$. There is an integer $N(p)$ and a holomorphic polynomial mapping g_p of degree $2p$ such that*

$$g_p : Q(2, 2p + 1) \to Q(N(p), 2p + 1).$$

and g_p maps to no hyperquadric with smaller numbers of positive or negative eigenvalues.

Proof. We begin with the group $\Gamma(2p, 2)$. We expand the formula given in (7) with p replaced by $2p$. The result is a polynomial $f_{2p,2}$ in the two variables x, y with the following properties. First, the coefficients are positive except for the coefficient of y^{2p} which is -1. Second, we have $f_{2p,2}(x, y) = 1$ on $x + y = 1$. Third, because of the group invariance, only even powers of x arise. We therefore can replace x

by $-x$ and obtain a polynomial $f(x,y)$ such that $f(x,y) = 1$ on $-x + y = 1$ and again, all coefficients are positive except for the coefficient of y^{2p}. Next replace y by $Y_1 + Y_2$. We obtain a polynomial in x, Y_1, Y_2 which has precisely $2p+1$ terms with negative coefficients. These terms arise from expanding $-(Y_1 + Y_2)^{2p}$. All other terms have positive coefficients. This polynomial takes the value 1 on the set $-x + Y_1 + Y_2 = 1$. Now replace x by $X_1 + \cdots + X_{2p+1}$.

We now have a polynomial $W(X,Y)$ that is 1 on the set given by

$$-\sum_{1}^{2p+1} X_j + \sum_{1}^{2} Y_j = 1.$$

It has precisely $2p+1$ terms with negative coefficients. There are many terms with positive coefficients; suppose that the number is $N(p)$. In order to get back to the holomorphic setting, we put $X_j = |z_j|^2$ for $1 \leq j \leq 2p+1$ and we put $Y_1 = |z_{2p+2}|^2$ and $Y_2 = |z_{2p+3}|^2$. We note that this idea (an example of the moment map) has been often used in this paper, as well as in the author's work on proper mappings between balls; see for example [D1] and [DKR]. Let $g_p(z)$ be the mapping, determined up to a diagonal unitary matrix, with

$$\sum_{j=1}^{N(p)} |g_j(z)|^2 - \sum_{j=1}^{2p+1} |g_j(z)|^2 = W(X,Y). \tag{26}$$

Each component of g_p is determined by (26) up to a complex number of modulus 1. The degree of g_p is the same as the degree of w. We obtain all the claimed properties. □

Example 6.1. We write out everything explicitly when $p = 1$. Let $c = \sqrt{2}$. The proof of Theorem 6.1 yields the polynomial mapping $g : Q(2,3) \to Q(8,3)$ of degree 2 defined by

$$g(z) = (z_1^2, z_2^2, z_3^2, cz_1z_2, cz_1z_3, cz_2z_3, cz_4, cz_5; z_4^2, cz_4z_5, z_5^2). \tag{27}$$

Notice that we used a semi-colon after the first eight terms to highlight that g maps to $Q(8,3)$. Summing the squared moduli of the first eight terms yields

$$(|z_1|^2 + |z_2|^2 + |z_3|^2)^2 + 2(|z_4|^2 + |z_5|^2). \tag{28}$$

Summing the squared moduli of the last three terms yields

$$(|z_4|^2 + |z_5|^2)^2. \tag{29}$$

The set $Q(2,3)$ is given by

$$|z_4|^2 + |z_5|^2 - 1 = |z_1|^2 + |z_2|^2 + |z_3|^2.$$

On this set we obtain 1 when we subtract (29) from (28).

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On the Subellipticity of Some Hypoelliptic Quasihomogeneous Systems of Complex Vector Fields

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In honor of Linda P. Rothschild

Abstract. For about twenty five years it was a kind of folk theorem that complex vector-fields defined on $\Omega \times \mathbb{R}_t$ (with Ω open set in $\mathbb{R}^n$) by

$$L_j = \frac{\partial}{\partial t_j} + i\frac{\partial \varphi}{\partial t_j}(\mathbf{t})\,\frac{\partial}{\partial x}\ ,\ j = 1, \dots, n\ ,\ \mathbf{t} \in \Omega, x \in \mathbb{R}\ ,$$

with φ analytic, were subelliptic as soon as they were hypoelliptic. This was indeed the case when $n = 1$ [Tr1] but in the case $n > 1$, an inaccurate reading of the proof (based on a non standard subelliptic estimate) given by Maire [Mai1] (see also Trèves [Tr2]) of the hypoellipticity of such systems, under the condition that φ does not admit any local maximum or minimum, was supporting the belief for this folk theorem. This question reappears in the book of [HeNi] in connection with the semi-classical analysis of Witten Laplacians. Quite recently, J.L. Journé and J.M. Trépreau [JoTre] show by explicit examples that there are very simple systems (with polynomial φ's) which were hypoelliptic but not subelliptic in the standard L^2-sense. But these operators are not quasihomogeneous.

In [De] and [DeHe] the homogeneous and the quasihomogeneous cases were analyzed in dimension 2. Large classes of systems for which subellipticiity can be proved were exhibited. We will show in this paper how a new idea for the construction of escaping rays permits to show that in the analytic case **all** the quasihomogeneous hypoelliptic systems in the class above considered by Maire are effectively subelliptic in the 2-dimensional case. The analysis presented here is a continuation of two previous works by the first author for the homogeneous case [De] and the two authors for the quasihomogeneous case [DeHe].

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1. Introduction and main result

1.1. Preliminaries on subellipticity and hypoellipticity

Let Ω an open set in $\mathbb{R}^n$ with $0 \in \Omega$ and $\varphi \in C^1(\Omega, \mathbb{R})$, with $\varphi(0) = 0$. We consider the regularity properties of the following family of complex vector fields on $\Omega \times \mathbb{R}$

$$(\mathbb{L}_\varphi) \quad L_j = \frac{\partial}{\partial t_j} + i \frac{\partial \varphi}{\partial t_j}(\mathbf{t}) \frac{\partial}{\partial x} \ , \ j = 1, \dots, n \ , \ \mathbf{t} \in \Omega, x \in \mathbb{R} \ , \tag{1.1}$$

We will concentrate our analysis near a point $(0, 0)$.

Many authors (among them [Ko2], [JoTre], [Mai1], [BDKT] and references therein) have considered this type of systems. For a given Ω, they were in particular interested in the existence, for some pair (s, N) such that $s+N > 0$, of the following family of inequalities.

For any pair of bounded open sets ω, I such that $\overline{\omega} \subset \Omega$ and $I \subset \mathbb{R}$, there exists a constant $C_{s,N}(\omega, I)$ such that

$$||u||_s^2 \leq C_N(\omega, I) \left(\sum_{j=1}^{n} ||L_j u||_0^2 + ||u||_{-N}^2 \right) \ , \ \forall u \in C_0^\infty(\omega \times I) \ , \tag{1.2}$$

where $|| \cdot ||_r$ denotes the Sobolev norm in $H^r(\Omega \times \mathbb{R})$.

If $s > 0$, we say that we have a subelliptic estimate and when φ is C^∞ this estimate is known to imply the hypoellipticity of the system. In [JoTre], there are also results where s can be arbitrarily negative.

The system (1.1) being elliptic in the $\mathbf{t}$ variable, it is enough to analyze the subellipticity microlocally near $\tau = 0$, i.e., near $(0, (0, \xi))$ in $(\omega \times I) \times (\mathbb{R}^{n+1} \setminus \{0\})$ with $\{\xi > 0\}$ or $\{\xi < 0\}$.

This leads to the analysis of the existence of two constants C_s^+ and C_s^- such that the two following semi-global inequalities hold:

$$\int_{\omega \times \mathbb{R}^+} \xi^{2s} |\widehat{u}(\mathbf{t}, \xi)|^2 \, dt d\xi \leq C_s^+ \int_{\omega \times \mathbb{R}^+} |\widehat{Lu}(\mathbf{t}, \xi)|^2 \, d\mathbf{t} d\xi \ , \ \forall u \in C_0^\infty(\omega \times \mathbb{R}) \ , \tag{1.3}$$

where $\widehat{u}(\mathbf{t}, \xi)$ is the partial Fourier transform of u with respect to the x variable, and

$$\int_{\omega \times \mathbb{R}^-} |\xi|^{2s} |\widehat{u}(\mathbf{t}, \xi)|^2 \, d\mathbf{t} d\xi \leq C_s^- \int_{\omega \times \mathbb{R}^-} |\widehat{Lu}(\mathbf{t}, \xi)|^2 \, d\mathbf{t} d\xi \ , \ \forall u \in C_0^\infty(\omega \times \mathbb{R}) \ . \tag{1.4}$$

When (1.3) is satisfied, we will speak of microlocal subellipticity in $\{\xi > 0\}$ and similarly when (1.4) is satisfied, we will speak of microlocal subellipticity in $\{\xi < 0\}$. Of course, when $s > 0$, it is standard that these two inequalities imply (1.2).

We now observe that (1.3) for φ is equivalent to (1.4) for $-\varphi$, so it is enough to concentrate our analysis on the first case.

1.2. The main results

As in [De] and [DeHe], we consider[1] φ in C^∞. We assume m and ℓ be given[2] in $\mathbb{R}^+$ such that

$$m \geq 2\ell \geq 2\ . \tag{1.5}$$

We consider in $\mathbb{R}^2$ (t,s) as the variables (instead of $\mathbf{t}$) and the functions $\varphi \in C^1(\mathbb{R}^2)$ will be $(1,\ell)$-quasihomogeneous of degree m in the following sense

$$\varphi(\lambda t, \lambda^\ell s) = \lambda^m \varphi(t,s)\ ,\ \forall (t,s,\lambda) \in \mathbb{R}^2 \times \mathbb{R}^+\ . \tag{1.6}$$

According to (1.6), the real function φ is determined by its restriction $\widetilde{\varphi}$ to the distorted circle $\mathcal{S}$

$$\widetilde{\varphi} := \varphi_{|\mathcal{S}}\ . \tag{1.7}$$

where $\mathcal{S}$ is defined by

$$\mathcal{S} = \{(t,s)\,;\, t^{2\ell} + s^2 = 1\}\ , \tag{1.8}$$

Our main results will be obtained under the following assumptions

Assumption 1.1 (Hnew1).

(i) $$\widetilde{\varphi}^{-1}(]0,+\infty[) \neq \emptyset\,.$$

(ii) *The zeroes of $\widetilde{\varphi}$ are not local maxima of $\widetilde{\varphi}$.*

Assumption 1.2 (Hnew2). *$\widetilde{\varphi} = \varphi_{/\mathcal{S}}$ has only a finite numbers of zeroes θ_j, each one being of finite order.*

Under this assumption, we denote by $p \geq 1$, the smallest integer such that, if θ_0 is any zero of $\widetilde{\varphi}$, there exists an integer $k \leq p$, such that

- if $\theta_0 \neq (0,\pm 1)$, then,
$$\widetilde{\varphi}^{(k)}(\theta_0) \neq 0\ , \tag{1.9}$$
- if $\theta_0 = (0,\pm 1)$,
$$|\varphi(t,s)| \geq \frac{1}{C}|t|^k\ , \tag{1.10}$$
for t close to 0.

Theorem 1.3. *Let $\varphi \in C^\infty$ be a $(1,\ell)$-quasihomogeneous function of order m (with (m,ℓ) satisfying (1.5)) and satisfying Assumptions 1.1 and 1.2. Then the associated system $\mathbb{L}_\varphi$ is microlocally ϵ-subelliptic in $\{\xi > 0\}$ with*

$$\frac{1}{\epsilon} = \sup(m,p)\ .$$

Theorem 1.4. *Let φ be a real analytic $(1,\ell)$-quasihomogeneous function of order m with $m \geq 2\ell \geq 2$.*
Under Assumption 1.1 the associated system $\mathbb{L}_\varphi$ is microlocally ϵ-subelliptic in $\{\xi > 0\}$ for some $\epsilon > 0$.

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[1]Like in these papers, the C^1 case could also be considered but the statements will be more complicate to formulate and we are mainly interested for this paper in the links between hypoellipticity and subellipticity.
[2]ℓ will be rational in the analytic case.

Theorem 1.5. *Let φ be a real analytic $(1,\ell)$-quasihomogeneous function of order m with $m \geq 2\ell \geq 2$.*
Then the system $\mathbb{L}_\varphi$ is hypoelliptic if and only if it is ϵ-subelliptic for some $\epsilon > 0$.

1.3. Comparison with previous results

We recall two classical theorems.

Theorem 1.6. *Let φ be C^∞ in a neighborhood of 0. Then if the system $\mathbb{L}_\varphi$ is microlocally hypoelliptic in $\{\xi > 0\}$, then there exists a neighborhood V of $0 \in \mathbb{R}^n$ such that φ has no local maximum in V*

This is Theorem III.1.1 in [Tr2] and we refer to [JoTre] for an elementary proof.

Theorem 1.7. *If φ is analytic without local maximum in a neighborhood of 0, then the system $\mathbb{L}_\varphi$ is (microlocally)-hypoelliptic in $\{\xi > 0\}$.*

We recall that H. Maire has shown in [Mai1] that the corresponding C^∞ statement of Theorem 1.7 is false when $n \geq 2$.

If φ does not admit any local maximum in a neighborhood of 0, and if φ is quasi-homogeneous of order m for some $m > 0$, then this implies that $\widetilde{\varphi}$ satisfies Assumption 1.1. So if we prove when φ is analytic, that this condition implies that the system is microlocally subelliptic in $\{\xi > 0\}$, we obtain immediately, using the necessary condition of Trèves ([Tr2], Theorem III.1.1), that, for quasihomogneous φ's, the system $\mathbb{L}_\varphi$ is microlocally hypoelliptic $\{\xi > 0\}$ if and only if the system is microlocally subelliptic $\{\xi > 0\}$.
We also obtain that the system $\mathbb{L}_\varphi$ is hypoelliptic if and only if the system is subelliptic. According to the counterexamples of Journé-Trépreau [JoTre], this cannot be improved.

The results in [De] (homogeneous case) and [DeHe] (quasihomogeneous case) were obtained under Assumptions (1.1) and (1.2), but with the additional condition:

Assumption 1.8 (Hadd).

(i) *If $\mathcal{S}_j^+$ is a connected component of $\widetilde{\varphi}^{(-1)}(]0,+\infty[)$, then one can write $\mathcal{S}_j^+$ as a finite union of arcs satisfying Property 1.9 below.*
(ii) *If $\mathcal{S}_j^-$ is one connected component of $\widetilde{\varphi}^{(-1)}(]-\infty,0[)$, then $\tilde{\varphi}$ has a unique minimum in $\mathcal{S}_j^-$.*

Here in the first item of Assumption 1.8, we mean by saying that a closed arc $[\theta,\theta']$ has Property 1.9 the following:

Property 1.9. *There exists on this arc a point $\widehat{\theta}$ such that:*

(a) *$\widetilde{\varphi}$ is non decreasing on the arc $\left[\theta,\widehat{\theta}\right]$ and non increasing on the arc $\left[\widehat{\theta},\theta'\right]$.*
(So the restriction of $\widetilde{\varphi}$ to $[\theta,\theta']$ has a maximum at $\widehat{\theta}$).
(b)
$$\langle \widehat{\theta} \,|\, \theta\rangle_\ell \geq 0 \text{ and } \langle \widehat{\theta} \,|\, \theta'\rangle_\ell \geq 0 \,, \tag{1.11}$$

where for $\theta = (\alpha, \beta)$ *and* $\widehat{\theta} = (\widehat{\alpha}, \widehat{\beta})$ *in* $\mathcal{S} \subset \mathbb{R}^2$,

$$\langle \widehat{\theta} \,|\, \theta \rangle_\ell := \widehat{\alpha}\alpha |\widehat{\alpha}|^{\ell-1} |\alpha|^{\ell-1} + \widehat{\beta}\beta \, . \tag{1.12}$$

Note here that we could have $\widetilde{\varphi}$ constant on $\mathcal{S}_{\theta,\theta'}$ and $\widehat{\theta} = \theta$ or θ'. Moreover item (b) says that the length of the two arcs is sufficiently small, more precisely that the distorted "angles" (see Section 3) associated to $[\theta, \theta']$ are acute.

Remarks 1.10.

(i) Assumption 1.8 was not so restrictive for the positive components of $\widetilde{\varphi}^{(-1)}(]0, +\infty[)$ (at least in the analytic case) but this was the condition (ii) on the uniqueness of the minimum which was introducing the most restrictive technical condition.

(ii) The proof of Theorem 1.3 consists in showing that Assumptions 1.1 and 1.2 imply Assumption $(H_+(\alpha))$, which was introduced in [De] and exploited in [DeHe] and which will be recalled in Section 2.

(iii) If φ is analytic and $\ell = \frac{\ell_2}{\ell_1}$ (with ℓ_1 and ℓ_2 mutually prime integers), all the criteria involving $\widetilde{\varphi}$ can be reinterpreted as criteria involving the restriction $\widehat{\varphi}$ of φ on

$$\mathcal{S}_{\ell_1,\ell_2} = \{(t, s) \, ; \, t^{2\ell_2} + s^{2\ell_1} = 1\} \, .$$

(iv) Due to Maire's characterization of hypoellipticity [Mai1] and Journé-Trépreau counterexamples our results are optimal.

Organization of the paper

As in [De] and [DeHe], the proof of our main theorem will be based on a rather "abstract" criterion established in [De], which will be recalled in Section 2. We recall the terminology adapted to the quasihomogeneity of the problem in Section 3. Section 4 will be devoted to the construction of escaping rays inside small sectors, which is the main novelty. Because a big part of the proof is based on the results of [DeHe], we will only emphasize in Section 5 on the new points of the proof permitting to eliminate all the finally non necessary assumptions of this previous paper.

2. Derridj's subellipticity criterion

We now recall the criterion established in [De]. This involves, for a given $\alpha > 0$, the following geometric escape condition on φ. We do not have in this section the restriction $n = 2$.

Assumption 2.1. $(H_+(\alpha))$

There exist open sets $\omega \subset \Omega$ *and* $\widetilde{\omega} \subset \omega$*, with full Lebesgue measure in* ω*, and a map* γ*:*

$$\widetilde{\omega} \times [0, 1] \ni (\mathbf{t}, \tau) \mapsto \gamma(\mathbf{t}, \tau) \in \Omega \, ,$$

such that

(i) $\gamma(\mathbf{t},0)=\mathbf{t}\,;\,\gamma(\mathbf{t},1)\notin\omega\,,\,\forall\mathbf{t}\in\widetilde{\omega}\,.$

(ii) *γ is of class C^1 outside a negligible set E and there exist $C_1>0$, $C_2>0$ and $C_3>0$ such that*

(a)
$$|\partial_\tau\gamma(\mathbf{t},\tau)|\leq C_2\,,\ \forall(\mathbf{t},\tau)\in\widetilde{\omega}\times[0,1]\setminus E\,.$$

(b)
$$|\det(D_{\mathbf{t}}\gamma)(\mathbf{t},\tau)|\geq\frac{1}{C_1}\,,\tag{2.1}$$

where $D_{\mathbf{t}}\gamma$ denotes the Jacobian matrix of γ considered as a map from $\widetilde{\omega}$ into $\mathbb{R}^2$.

(c)
$$\varphi(\gamma(\mathbf{t},\tau))-\varphi(\mathbf{t})\geq\frac{1}{C_3}\tau^\alpha\,,\ \forall(\mathbf{t},\tau)\in\widetilde{\omega}\times[0,1]\,.$$

Using this assumption, it is proved in [De] the following theorem.

Theorem 2.2.

– *If φ satisfies $(H_+(\alpha))$, then the associated system $(1.1)_\varphi$ is microlocally $\frac{1}{\alpha}$-subelliptic in $\{\xi>0\}$.*
– *If $-\varphi$ satisfies $(H_+(\alpha))$, then the associated system $(1.1)_\varphi$ is microlocally $\frac{1}{\alpha}$-subelliptic in $\{\xi<0\}$.*
– *If φ and $-\varphi$ satisfy $(H_+(\alpha))$, then the associated system $(1.1)_\varphi$ is $\frac{1}{\alpha}$-subelliptic.*

3. Quasihomogeneous structure

3.1. Distorted geometry

Condition (i) in Assumption 2.1 expresses the property that the curve is escaping from ω. For the description of escaping curves, it appears useful to extend the usual terminology used in the Euclidean space $\mathbb{R}^2$ in a way which is adapted to the given quasihomogeneous structure. This is realized by introducing the *dressing* map:

$$(t,s)\mapsto d_\ell(t,s)=\left(t\,|t|^{\ell-1},s\right)\,.\tag{3.1}$$

which is at least of class C^1 as $\ell\geq1$, and whose main role is to transport the distorted geometry onto the Euclidean geometry.
The first example was the unit distorted circle (in short unit disto-circle or unit "circle") $\mathcal{S}$ introduced in (1.8) whose image by d_ℓ becomes the standard unit circle in $\mathbb{R}^2$ centered at $(0,0)$.
Similarly, we will speak of disto-sectors, disto-arcs, disto-rays, disto-disks. In particular, for $(a,b)\in\mathcal{S}$, we define the disto-ray $\mathcal{R}_{(a,b)}$ by

$$\mathcal{R}_{(a,b)}:=\{(\lambda a,\lambda^\ell b)\,;\,0\leq\lambda\leq1\}\,.\tag{3.2}$$

The disto-scalar product of two vectors in $\mathbb{R}^2$ (t,s) et (t',s') is then given by

$$\langle(t,s)\mid(t',s')\rangle_\ell=tt'|tt'|^{\ell-1}+ss'\,.\tag{3.3}$$

(for $\ell=1$, we recover the standard scalar product).

For $(t,s) \in \mathbb{R}^2$, we introduce also the quasihomogeneous positive function ϱ defined on $\mathbb{R}^2$ by:

$$\varrho(t,s)^{2\ell} = t^{2\ell} + s^2 \,. \tag{3.4}$$

With this notation, we observe that, if $(t,s) \in \mathbb{R}^2 \setminus \{(0,0)\}$, then

$$(\widetilde{t}\,,\,\widetilde{s}) := \left(\frac{t}{\varrho(t,s)}\,,\,\frac{s}{\varrho(t,s)}\right) \in \mathcal{S}\,, \tag{3.5}$$

and

$$(t,s) \in \mathcal{R}_{(\widetilde{t},\widetilde{s})}\,.$$

The open disto-disk $D(R)$ is then defined by

$$D(R) = \{(x,y) \mid \varrho(x,y) < R\}\,. \tag{3.6}$$

We can also consider a parametrization of the disto-circle by a parameter on the corresponding circle $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ (through the dressing map). We note that we have a natural (anticlockwise) orientation of the disto-circle. In other cases it will be better to parametrize by s (if $t \neq 0$) or by t (if $s \neq 0$). So a point in $\mathcal{S}$ will be defined either by θ or by $(a,b) \in \mathbb{R}^2$ or by ϑ.

Once an orientation is defined on $\mathcal{S}$, two points θ_1 and θ_2 (or (a_1,b_1) and (a_2,b_2)) on $\mathcal{S}$ will determine a unique unit "sector" $S(\theta_1,\theta_2) \subset D(1)$.

3.2. Distorted dynamics

The parametrized curves γ permitting us to satisfy Assumption 2.1 will actually be "lines" (possibly broken) which will finally escape from a neighborhood of the origin. Our aim in this subsection is to define these "lines" (actually distorted parametrized lines).
In parametric coordinates, with

$$t(\tau) = t + \nu\,\tau\,, \tag{3.7}$$

the curve γ starting from (t,s) and disto-parallel to (c,d) is defined by writing that the vector $(t(\tau)|t(\tau)|^{\ell-1} - t|t|^{\ell-1}, s(\tau)-s)$ is parallel to $(c|c|^{\ell-1}, d)$.
In the applications, we will only consider $\nu = \pm c$ and $c \neq 0$.
So

$$\left(t(\tau)|t(\tau)|^{\ell-1} - t|t|^{\ell-1}\right) d = c|c|^{\ell-1}(s(\tau)-s)\,,$$

and we find

$$s(\tau) = s + \frac{d}{c|c|^{\ell-1}}\left(t(\tau)|t(\tau)|^{\ell-1} - t|t|^{\ell-1}\right)\,, \tag{3.8}$$

We consider the map $\sigma \mapsto f_\ell(\sigma)$ which is defined by

$$f_\ell(\sigma) = \sigma|\sigma|^{\ell-1}\,. \tag{3.9}$$

Note that

$$f_\ell'(\sigma) = \ell|\sigma|^{\ell-1} \geq 0\,. \tag{3.10}$$

With this new function, (3.8) can be written as

$$df_\ell(t(\tau)) - s(\tau)f_\ell(c) = df_\ell(t) - sf_\ell(c)\,. \tag{3.11}$$

This leads us to use the notion of distorted determinant of two vectors in $\mathbb{R}^2$. For two vectors $v := (v_1, v_2)$ and $w := (w_1, w_2)$, we introduce:

$$\Delta_\ell(v; w) = \Delta_\ell(v_1, v_2, w_1, w_2) := f_\ell(v_1)w_2 - v_2 f_\ell(w_1) \; . \tag{3.12}$$

With this notation, (3.11) can be written

$$\Delta_\ell((c,d); (t(\tau), s(\tau))) = \Delta_\ell(c, d, t(\tau), s(\tau)) = \Delta_\ell(c, d, t, s) \; , \tag{3.13}$$

When $\ell = 1$, we recover the usual determinant of two vectors in $\mathbb{R}^2$. We now look at the variation of ψ which is defined (for a given initial point (t, s)) by

$$\tau \mapsto \psi(\tau) = \rho(\tau)^{2\ell} = t(\tau)^{2\ell} + s(\tau)^2 \; . \tag{3.14}$$

We now need the following lemma proved in [DeHe].

Lemma 3.1. *Under Condition*

$$c\nu > 0 \; , \; \langle (c,d) \mid (s,t) \rangle_\ell > 0 \; , \tag{3.15}$$

we have, for any $\tau \geq 0$, for any $(t, s) \in \mathbb{R}^2 \setminus (0, 0)$

$$\rho(\tau)^{2\ell} - \rho(0)^{2\ell} \geq (\frac{\nu \tau}{2c})^{2\ell} \; . \tag{3.16}$$

If instead $c\nu < 0$, we obtain:

$$\rho(\tau)^{2\ell} - \rho(0)^{2\ell} \leq -(\frac{\nu \tau}{2c})^{2\ell} \; . \tag{3.17}$$

We continue by analyzing the variation of $s(\tau)$ and $t(\tau)$ and more precisely the variation on the disto-circle of:

$$\widetilde{t}(\tau) = \frac{t(\tau)}{\rho(\tau)} \; , \; \widetilde{s}(\tau) = \frac{s(\tau)}{\rho(\tau)^\ell} \; . \tag{3.18}$$

After some computations, we get, with

$$\nu = \pm c \; ,$$

$$\widetilde{t}'(\tau) = \pm |c|^{1-\ell} \frac{s(\tau)}{\rho(\tau)^{2\ell+1}} \Delta_\ell(c, d, t, s) \; , \tag{3.19}$$

which can also be written in the form

$$\widetilde{t}'(\tau) = \pm |c|^{1-\ell} \frac{\widetilde{s}(\tau)}{\rho(\tau)} \Delta_\ell(c, d, \widetilde{t}(\tau), \widetilde{s}(\tau)) \; . \tag{3.20}$$

Similarly, we get for $\widetilde{s}'$,

$$\widetilde{s}'(\tau) = \mp \ell |c|^{1-\ell} \frac{t(\tau)^{2\ell-1}}{\rho(\tau)^{3\ell}} \Delta_\ell(c, d, t, s) \; , \tag{3.21}$$

and

$$\widetilde{s}'(\tau) = \mp \ell |c|^{1-\ell} \frac{\widetilde{t}^{2\ell-1}(\tau)}{\rho(\tau)} \Delta_\ell(c, d, \widetilde{t}(\tau), \widetilde{s}(\tau)) \; . \tag{3.22}$$

4. Analysis of the quasielliptic case ($\widetilde{\varphi} > 0$)

We first start the proof of the main theorem with the particular case when

$$\widetilde{\varphi} \geq \mu > 0 \,. \tag{4.1}$$

This case is already interesting for presenting the main ingredients of the general proof. We can remark indeed that what we are doing below in $\mathcal{S}$ can be done later in a specific (disto)-arc of $\mathcal{S}$. In addition, it will be applied ((with reverse time) in the region where $\widetilde{\varphi}$ is negative.

4.1. Construction of γ

We construct γ for points (s,t) belonging to a unit sector $S(\theta_1,\theta_2)$ associated to some arc (θ_1,θ_2). For a given pair (c,d) in $\mathcal{S}$, in a sufficiently small neighborhood of (θ_1,θ_2), we now define γ (see (3.7)-(3.8), with $\nu = c \neq 0$) by

$$\gamma(t,s,\tau) := (t(\tau),s(\tau)) := (t + c\tau, s + \frac{d}{f_\ell(c)}(f_\ell(t(\tau)) - f_\ell(t)) \,. \tag{4.2}$$

Remark 4.1. Note that for any (t_0,s_0,τ) the Jacobian of the map $(t,s) \mapsto \gamma(t,s,\tau)$ at (t_0,s_0) is 1.

4.2. Analysis of $\rho(\tau)^m - \rho^m$

Using (3.16) and

$$\rho(\tau)^m - \rho^m = \left(\rho(\tau)^{2\ell}\right)^{\frac{m}{2\ell}} - \left(\rho^{2\ell}\right)^{\frac{m}{2\ell}} \geq \left(\rho(\tau)^{2\ell} - \rho^{2\ell}\right)^{\frac{m}{2\ell}} \,,$$

where we note that $m \geq 2\ell$, we deduce

$$\rho(\tau)^m - \rho^m \geq 2^{-m}\tau^m \,,\ \forall \tau \geq 0 \,. \tag{4.3}$$

A second trivial estimate, will be useful:

$$\rho(\tau)^m - \rho^m \geq [\rho(\tau)^{2\ell} - \rho^{2\ell}]\rho^{m-2\ell} \,. \tag{4.4}$$

4.3. The lower bound in the quasi-homogeneous case

Lemma 4.2. *There exist for any $\mu > 0$, $\delta(\mu) > 0$ and $\epsilon(\mu)$ such that, for any truncated sector $S(\theta_1,\theta_2)$ attached to an arc (θ_1,θ_2) contained in $\widetilde{\varphi}^{(-1)}(\mu,+\infty[)$ of openness less than $\delta(\mu)$ and any pair (c,d) inside an $\epsilon(\mu)$-neighborhood of (θ_1,θ_2), with $c \neq 0$, then, for all $(t,s) \in S(\theta_1,\theta_2)$, all $\tau \geq 0$ such that $(t(\tau),s(\tau)) \in S(\theta_1,\theta_2)$, there exists $C > 0$ such that*

$$\varphi(t(\tau),s(\tau)) - \varphi(t,s) \geq \frac{1}{C}\tau^m \,. \tag{4.5}$$

We start with the case

$$m = 2\ell \,,$$

and will use the following decomposition for estimating from below the variation $\varphi(t(\tau),s(\tau)) - \varphi(t,s)$.

$$\varphi(t(\tau),s(\tau)) - \varphi(t,s) = (\rho(\tau)^{2\ell} - \rho^{2\ell})\widetilde{\varphi}(\theta_\tau) + \rho^{2\ell}(\widetilde{\varphi}(\theta_\tau) - \widetilde{\varphi}(\theta)) = (I) + (II) \,. \tag{4.6}$$

So it is enough to show

$$\begin{cases} |II| \leq \frac{I}{2} \,, \\ I \geq \frac{1}{C}\tau^{2\ell} \,, \end{cases} \tag{4.7}$$

for getting the right estimate for $I + II$:

$$I + II \geq \frac{1}{C}\tau^{2\ell} \,. \tag{4.8}$$

Take a small arc $S(\theta_1, \theta_2)$ on which

$$\widetilde{\varphi} \geq \mu > 0 \,.$$

We consider, starting from (s,t) in $S(\theta_1, \theta_2)$, a positive "half-ray" $\gamma(t,s,\tau)$ with direction (c,d) in an ϵ- neighborhood of (θ_1, θ_2) (inside or outside of (θ_1, θ_2) in $\mathcal{S}$).

Remark 4.3. We emphasize that the choice of (c,d) is relatively free (we do not have used actually that (c,d) is at a maximum point of $\widetilde{\varphi}$ like in [DeHe]). It is enough that (c,d) is in a sufficiently small neighborhood of (θ_1, θ_2) (inside or outside) and **both cases** will be used at the end.

Note that with

$$\psi(\tau) = \rho(\tau)^{2\ell} - \rho^{2\ell} \,,$$

we have

$$\psi'(\tau) = 2\ell |\frac{t(\tau)}{c}|^{\ell-1} \langle (c,d), (t(\tau), s(\tau)) \rangle_\ell \,. \tag{4.9}$$

We will only estimate I and II for $(t(\tau), s(\tau))$ inside $S((\theta_1, \theta_2)$ and $\tau \leq 1$. This could be an effective restriction when (c,d) does not belong to (θ_1, θ_2).
We observe that when the arc is of sufficiently small angle (measured by δ), then, on this part of the curve, we have

$$\langle (c,d) \,,\ (t(\tau), s(\tau)) \rangle_\ell \geq (1-\delta)\rho^\ell \,. \tag{4.10}$$

We now distinguish two cases. The first one corresponds to the case when $S(\theta_1, \theta_2)$ is far from $(0, \pm 1)$ and the second case corresponds to the case when $S(\theta_1, \theta_2)$ is close to $(0, \pm 1)$.

First case. We will assume that

$$|\widetilde{t}(\tau)| \geq \frac{1}{4} > 0 \ \text{ and } |c| \geq \frac{1}{4} \,. \tag{4.11}$$

We then deduce from (4.9), (4.10) and (4.11) the following lower bound

$$\psi'(\tau) \geq \frac{1}{C}\,(1-\delta)\rho^{2\ell-1}. \tag{4.12}$$

Now we have

$$|\widetilde{\varphi}(\theta_\tau) - \widetilde{\varphi}(\theta)| \leq C|\theta_\tau - \theta| \leq \tilde{C}\left(|\tilde{t}(\tau) - \tilde{t}| + |\tilde{s}(\tau) - \tilde{s}|\right)$$

We will show, that if the angle of the sector is less than δ then

$$\rho^{2\ell}\left(|\tilde{s}'(\tau)| + |\tilde{t}'(\tau)|\right) \leq \epsilon(\delta)\psi'(\tau) \,, \tag{4.13}$$

where $\epsilon(\delta)$ tends to 0 as $\delta \to 0$.

This last inequality is immediate from (4.12), (3.22) and (3.20).

This leads by integration to

$$II \leq \epsilon(\delta) I \; . \tag{4.14}$$

Second case. We now suppose that

$$|\widetilde{t}(\tau)| \leq \frac{1}{2} \text{ and } |c| \leq \frac{1}{2} \; . \tag{4.15}$$

In this case, we have only

$$\psi'(\tau) \geq \frac{1}{C}\, |c|^{1-\ell}(1-\delta)|\tilde{t}(\tau)|^{\ell-1}\rho^{2\ell-1}. \tag{4.16}$$

On the other hand, we can parametrize $\mathcal{S}$ by $\tilde{t}$ and we get

$$\chi(\tilde{t}) := \widetilde{\varphi}(\tilde{t},\tilde{s}) = \varphi(\tilde{t},(1-\tilde{t}^{2\ell})^{\frac{1}{2}}) \; .$$

So let us consider

$$\kappa(\tau) = \chi(\tilde{t}(\tau)) \; .$$

We get

$$\kappa'(\tau) = \tilde{t}'(\tau)\chi'(\tilde{t}(\tau)) =$$
$$= \tilde{t}'(\tau)\left((\partial_t\varphi)(\tilde{t}(\tau),(1-\tilde{t}(\tau)^{2\ell})^{\frac{1}{2}}) - \ell\tilde{t}(\tau)^{2\ell-1}(1-\tilde{t}^{2\ell})^{-\frac{1}{2}}(\partial_s\varphi)(\tilde{t},(1-\tilde{t}^{2\ell})^{\frac{1}{2}})\right) .$$

Using the property that φ satisfies (A.2) and that (4.15) and (3.20) are satisfied, we obtain the existence of $C>0$ and $\widehat{C}$ such that

$$|\kappa'(\tau)| \leq C|\tilde{t}'(\tau)|\left(|\tilde{t}(\tau)|^{\ell-1} + |\tilde{t}|^{2\ell-1}\right) \leq \widehat{C}\,\frac{1}{\rho}\,|c|^{1-\ell}\,|\tilde{t}(\tau)|^{\ell-1} \; . \tag{4.17}$$

Moreover (3.20) shows that $\widehat{C}$ tends to zero as the openness of the sector tends to zero.

We can achieve the proof of this case by comparing the upper bound of $\rho^{2\ell}\kappa'(\tau)$ and the lower bound of $\psi'(\tau)$.

Remark 4.4. Till now, we have only treated the case $m=2\ell$. For the general case, it is enough to use the inequality (4.4).

Remark 4.5. For the lower bound in the second line of (4.7), we can use (4.3), which is proved under the weak condition (3.15), which will be satisfied in our case because we consider sufficiently small sectors.

4.4. The case of arcs in $\widetilde{\varphi} \geq 0$ but with a zero at one end

So we consider a closed arc (θ_0,θ_1) who has a zero of $\widetilde{\varphi}$ at one end, say θ_0. Then, Assumption 1.2 implies that if the opening of this arc is sufficiently small $\widetilde{\varphi}$ is strictly increasing. We are actually in a situation that we have considered in [DeHe]. We have from Section 5 in [DeHe]

Lemma 4.6. *There exists δ_0 such that if the openness of (θ_0, θ_1) is smaller than δ_0, then*

$$\varphi(t(\tau), s(\tau)) - \varphi(t,s)) \geq c_0 \tau^{\sup(m,p)} \tag{4.18}$$

for $\tau \geq 0$ and any $(t,s) \in S(\theta_1, \theta_2)$.

Remark 4.7. Actually this analysis will be mainly applied in reverse time in the negative zone for $\widetilde{\varphi}$. This is indeed here that the assumptions in [DeHe] were unnecessarily too strong. So we need the estimates (when taking an exit direction which is exterior to the sector) for the part of the trajectory which remains inside the sector.

5. Completion of the proof

Let us be more precise on how we can put the arguments together. The zeroes x_j $(j = 1, \dots, k)$ of $\widetilde{\varphi}$ which are isolated by our assumptions determine disjoint arcs (x_j, x_{j+1}) on which $\widetilde{\varphi}$ has a constant sign. We call "negative" arc an arc on which $\widetilde{\varphi}$ is strictly negative in its interior.
Our assumptions imply that two negative arcs can not touch and that there exists at least a positive arc.
Associated to each arc, we have a natural sector and a natural truncated sector (the sector intersected with a ball of radius less than 1 which will be shown as uniformly bounded from below by a strictly positive constant). For each of these sectors, we have to determine how we can escape starting from the truncated sector.

The case of the "positive arcs" has been essentially treated in our previous paper. The new arguments can give a partially alternative solution and actually improvements: outside a neighborhood of the zeros, we do not need any assumptions in the C^∞ case.

The case of the "negative arcs" is more interesting because this was in this case that we meet unnecessary conditions.

Let (θ_0, θ_1) such an arc and $S(\theta_0, \theta_1)$ the associated sector. So we have

$$\theta_0 < \theta_1 \ , \ \widetilde{\varphi}(\theta_0) = 0 = \widetilde{\varphi}(\theta_1) \ .$$

In addition, we have

$$\widetilde{\varphi}(\theta) < 0 \ , \ \forall \theta \in]\theta_0, \theta_1[\ .$$

According to our assumptions, $\widetilde{\varphi}$ changes of sign at θ_0 and θ_1.

We now construct a finite sequence ω_j such that

$$\theta_0 < \omega_1 < \omega_2 < \cdots < \omega_L < \theta_1 \ ,$$

and such that

- One can escape from the truncated sectors $S(\theta_0, \omega_1)$ and $S(\omega_L, \theta_1)$ by the method presented in our previous paper [DeHe] (this is possible by choosing ω_1 sufficiently close to θ_0 and ω_L sufficiently close to θ_1).

- For the other arcs (ω_j, ω_{j+1}) we construct escaping rays permitting to touch the neighboring sector $S(\omega_{j+1}, \omega_{j+2})$ (with the convention that $\omega_{L+1} = \theta_1$). So iterating at most L times we will arrive to a positive sector.

So we are reduced, modulo what we have done in the previous paper, to control the situation in a strictly negative subsector $S(\omega_j, \omega_{j+1})$ and this problem is solved under the condition that $|\omega_j - \omega_{j+1}|$ is small enough.

One choose indeed a point $\omega_{j-1} < \omega'_j < \omega_j$ and we will escape of the sector $S(\omega_j, \omega_{j+1})$ by considering the curve

$$\gamma(t, s, \tau) = \left(t - c_j\tau, s - \frac{d_j}{f_\ell(c_j)} \left(f_\ell(t(\tau)) - f_\ell(t)\right) \right) ,$$

with $\omega'_j = (c_j, d_j)$.

Reversing the time, we can apply the results of Section 4 to $-\widetilde{\varphi}$ in the sectors where $\widetilde{\varphi}$ is strictly negative.

The only new point is that we have to work with sufficiently small sectors (because of the condition appearing in Lemma 4.2). This imposes possibly a larger but finite number of broken lines. The control of the Jacobians in Section 6 of [DeHe] is exactly the same.

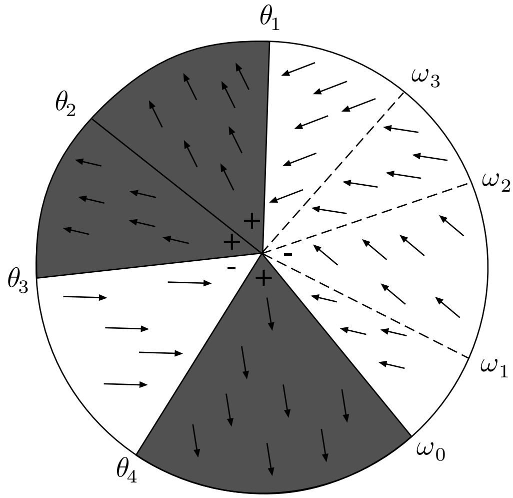

FIGURE 1. The escaping flow (after application of the dressing map)

Appendix A. A technical proposition

Proposition A.1. *Let $\varphi \in C^\infty$ be a $(1, \ell)$-quasihomogeneous function of degree m, with $\ell \geq 1$ and suppose that*

$$\varphi(0, 1) \neq 0 . \tag{A.1}$$

Then, there exists a C^∞ function g such that

$$\varphi(t,s) = \varphi(0,s) + t^\ell g(t,s)\ , \quad \textit{for } s>0\ . \tag{A.2}$$

Remark A.2. In the analytic case, we now assume that

$$\ell = \ell_2/\ell_1\ , \tag{A.3}$$

with ℓ_1 and ℓ_2 mutually prime integers. In this case, the quasihomogeneity Assumption (1.6) on φ implies that φ is actually a polynomial and we can write φ in the form

$$\varphi(t,s) = \sum_{\ell_1 j + \ell_2 k = \ell_1 m} a_{j,k} t^j s^k\ , \tag{A.4}$$

where (j,k) are integers and the $a_{j,k}$ are real. It results of (A.4), that under the hypothesis A.1 one has $\ell_2 k = \ell_1 m$ so $\frac{m}{\ell}$ is an integer and $a_{0,\frac{m}{\ell}} \neq 0$. Moreover, the j's appearing in the sum giving (A.4) satisfy $\frac{j}{\ell}$ is an integer. This shows in particular (A.2).

Proof. The proof in the C^∞ case is quite close to the analytic case. Using the quasihomogeneity, we have, for $s>0$,

$$\varphi(t,s) = s^{\frac{m}{\ell}} \varphi(s^{-\frac{1}{\ell}} t, 1)\ .$$

Differentiating k-times with respect to t and taking $t=0$, we get

$$(\partial_t^k \varphi)(0,s) = s^{\frac{m-k}{\ell}} (\partial_t^k \varphi)(0,1)\ . \tag{A.5}$$

The right-hand side is C^∞ under the condition that $(\partial_t^k\varphi)(0,1)=0$ for $\frac{m-k}{\ell} \notin \mathbb{N}$.
For $k=0$, we obtain under our assumption that $\frac{m}{\ell}$ should be an integer.
Then we obtain, using this last property, that $(\partial_t^k\varphi)(0,1)=0$ for $k=1,\dots,\ell-1$, hence

$$(\partial_t^k\varphi)(0,s) = 0\ , \quad \text{for } k=1,\dots,\ell-1\ .$$

This gives immediately the proposition.

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Invariance of the Parametric Oka Property

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Dedicated to Linda P. Rothschild

Abstract. Assume that E and B are complex manifolds and that $\pi\colon E \to B$ is a holomorphic Serre fibration such that E admits a finite dominating family of holomorphic fiber-sprays over a small neighborhood of any point in B. We show that the parametric Oka property (POP) of B implies POP of E; conversely, POP of E implies POP of B for contractible parameter spaces. This follows from a parametric Oka principle for holomorphic liftings which we establish in the paper.

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1. Oka properties

The main result of this paper is that a subelliptic holomorphic submersion $\pi\colon E \to B$ between (reduced, paracompact) complex spaces satisfies the *parametric Oka property*. *Subellipticity* means that E admits a finite dominating family of holomorphic fiber-sprays over a neighborhood of any point in B (Def. 2.3). The conclusion means that for any Stein source space X, any compact Hausdorff space P (the parameter space), and any continuous map $f\colon X \times P \to B$ which is X-holomorphic (i.e., such that $f_p = f(\cdot\,, p)\colon X \to B$ is holomorphic for every $p \in P$), a continuous lifting $F\colon X \times P \to E$ of f (satisfying $\pi \circ F = f$) can be homotopically deformed through liftings of f to an X-holomorphic lifting. (See Theorem 4.2 for a precise statement.)

$$\begin{array}{ccc} & & E \\ & \overset{F}{\nearrow} & \big\downarrow{\scriptstyle \pi} \\ X \times P & \xrightarrow{\;f\;} & B \end{array}$$

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The following result is an easy consequence. Suppose that E and B are complex manifolds and that $\pi\colon E \to B$ is a subelliptic submersion which is also a Serre fibration (such map is called a *subelliptic Serre fibration*), or is a holomorphic fiber bundle whose fiber satisfies the parametric Oka property. Then the parametric Oka property passes up from the base B to the total space E; it also passes down from E to B if the parameter space P is contractible, or if π is a weak homotopy equivalence (Theorem 1.2).

We begin by recalling the relevant notions. Among the most interesting phenomena in complex geometry are, on the one hand, *holomorphic rigidity*, commonly expressed by Kobayashi-Eisenman hyperbolicity; and, on the other hand, *holomorphic flexibility*, a term introduced in [7]. While Kobayashi hyperbolicity of a complex manifold Y implies in particular that there exist no nonconstant holomorphic maps $\mathbb{C} \to Y$, flexibility of Y means that it admits many nontrivial holomorphic maps $X \to Y$ from any Stein manifold X; in particular, from any Euclidean space $\mathbb{C}^n$.

The most natural flexibility properties are the *Oka properties* which originate in the seminal works of Oka [27] and Grauert [14, 15]. The essence of the classical *Oka-Grauert principle* is that a complex Lie group, or a complex homogeneous manifold, Y, enjoys the following:

Basic Oka Property (BOP) of Y: *Every continuous map $f\colon X \to Y$ from a Stein space X is homotopic to a holomorphic map. If in addition f is holomorphic on (a neighborhood of) a compact $\mathcal{O}(X)$-convex subset K of X, and if $f|_{X'}$ is holomorphic on a closed complex subvariety X' of X, then there is a homotopy $f^t\colon X \to Y$ $(t \in [0,1])$ from $f^0 = f$ to a holomorphic map f^1 such that for every $t \in [0,1]$, f^t is holomorphic and uniformly close to f^0 on K, and $f^t|_{X'} = f|_{X'}$.*

All complex spaces in this paper are assumed to be reduced and paracompact. A map is said to be holomorphic on a compact subset K of a complex space X if it is holomorphic in an open neighborhood of K in X; two such maps are identified if they agree in a (smaller) neighborhood of K; for a family of maps, the neighborhood should be independent of the parameter.

When $Y = \mathbb{C}$, BOP combines the Oka-Weil approximation theorem and the Cartan extension theorem. BOP of Y means that, up to a homotopy obstruction, the same approximation-extension result holds for holomorphic maps $X \to Y$ from any Stein space X to Y.

Denote by $\mathcal{C}(X,Y)$ (resp. by $\mathcal{O}(X,Y)$) the space of all continuous (resp. holomorphic) maps $X \to Y$, endowed with the topology of uniform convergence on compacts. We have a natural inclusion

$$\mathcal{O}(X,Y) \hookrightarrow \mathcal{C}(X,Y). \tag{1.1}$$

BOP of Y implies that every connected component of $\mathcal{C}(X,Y)$ contains a component of $\mathcal{O}(X,Y)$. By [8, Theorem 5.3], BOP also implies the following

One-parametric Oka Property: *A path $f\colon [0,1] \to \mathcal{C}(X,Y)$ such that $f(0)$ and $f(1)$ belong to $\mathcal{O}(X,Y)$ can be deformed, with fixed ends at $t = 0, 1$, to a path in*

$\mathcal{O}(X,Y)$. Hence (1.1) *induces a bijection of the path connected components of the two spaces.*

Y enjoys the **Weak Parametric Oka Property** if for each finite polyhedron P and subpolyhedron $P_0 \subset P$, a map $f\colon P \to \mathcal{C}(X,Y)$ such that $f(P_0) \subset \mathcal{O}(X,Y)$ can be deformed to a map $\widetilde{f}\colon P \to \mathcal{O}(X,Y)$ by a homotopy that is fixed on P_0:

$$\begin{array}{ccc} P_0 & \longrightarrow & \mathcal{O}(X,Y) \\ {\scriptstyle incl}\downarrow & \overset{\widetilde{f}}{\nearrow} & \downarrow{\scriptstyle incl} \\ P & \xrightarrow{\ f\ } & \mathcal{C}(X,Y) \end{array}$$

This implies that (1.1) is a weak homotopy equivalence [11, Corollary 1.5].

Definition 1.1. (Parametric Oka Property (POP)) Assume that P is a compact Hausdorff space and that P_0 is a closed subset of P. A complex manifold Y enjoys POP for the pair (P, P_0) if the following holds. Assume that X is a Stein space, K is a compact $\mathcal{O}(X)$-convex subset of X, X' is a closed complex subvariety of X, and $f\colon X \times P \to Y$ is a continuous map such that

(a) the map $f_p = f(\cdot\,,p)\colon X \to Y$ is holomorphic for every $p \in P_0$, and
(b) f_p is holomorphic on $K \cup X'$ for every $p \in P$.

Then there is a homotopy $f^t\colon X \times P \to Y$ $(t \in [0,1])$ such that f^t satisfies properties (a) and (b) above for all $t \in [0,1]$, and also

(i) f^1_p is holomorphic on X for all $p \in P$,
(ii) f^t is uniformly close to f on $K \times P$ for all $t \in [0,1]$, and
(iii) $f^t = f$ on $(X \times P_0) \cup (X' \times P)$ for all $t \in [0,1]$.

The manifold Y satisfies POP if the above holds for each pair $P_0 \subset P$ of compact Hausdorff spaces. Analogously we define POP for sections of a holomorphic map $Z \to X$. □

Restricting POP to pairs $P_0 \subset P$ consisting of finite polyhedra we get Gromov's Ell_∞ property [16, Def. 3.1.A.]. By Grauert, all complex homogeneous manifolds enjoy POP for finite polyhedral inclusions $P_0 \subset P$ [14, 15]. A weaker sufficient condition, called *ellipticity* (the existence of a dominating spray on Y, Def. 2.1 below), was found by Gromov [16]. A presumably even weaker condition, *subellipticity* (Def. 2.2), was introduced in [4].

If Y enjoys BOP or POP, then the corresponding Oka property also holds for sections of any holomorphic fiber bundle $Z \to X$ with fiber Y over a Stein space X [10]. See also Sect. 2 below and the papers [5, 21, 22, 23].

It is important to know which operations preserve Oka properties. The following result was stated in [8] (remarks following Theorem 5.1), and more explicitly in [9, Corollary 6.2]. (See also [16, Corollary 3.3.C'].)

Theorem 1.2. *Assume that E and B are complex manifolds. If $\pi\colon E \to B$ is a subelliptic Serre fibration* (*Def.* 2.3 *below*)*, or a holomorphic fiber bundle with* POP *fiber, then the following hold:*

(i) *If B enjoys the parametric Oka property* (POP)*, then so does E.*
(ii) *If E enjoys* POP *for contractible parameter spaces P* (*and arbitrary closed subspaces P_0 of P*)*, then so does B.*
(iii) *If in addition $\pi\colon E \to B$ is a weak homotopy equivalence then*

$$\text{POP of } E \Longrightarrow \text{POP of B.}$$

All stated implications hold for a specific pair $P_0 \subset P$ of parameter spaces.

The proof Theorem 1.2, proposed in [9], requires the parametric Oka property for sections of certain continuous families of subelliptic submersions. When Finnur Lárusson asked for explanation and at the same time told me of his applications of this result [24] (personal communication, December 2008), I decided to write a more complete exposition. We prove Theorem 1.2 in Sec. 5 as a consequence of Theorem 4.2. This result should be compared with Lárusson's [24, Theorem 3] where the map $\pi\colon E \to B$ is assumed to be an *intermediate fibration* in the model category that he constructed.

Corollary 1.3. *Let $Y = Y_m \to Y_{m-1} \to \cdots \to Y_0$, where each Y_j is a complex manifold and every map $\pi_j\colon Y_j \to Y_{j-1}$ $(j = 1, 2, \ldots, m)$ is a subelliptic Serre fibration, or a holomorphic fiber bundle with* POP *fiber. Then the following hold:*
(i) *If one of the manifolds Y_j enjoys* BOP*, or* POP *with a contractible parameter space, then all of them do.*
(ii) *If in addition every π_j is acyclic* (*a weak homotopy equivalence*) *and if Y is a Stein manifold, then every manifold Y_j in the tower satisfies the implication* BOP $\Longrightarrow$ POP*.*

Proof. Part (i) is an immediate consequences of Theorem 1.2. For (ii), observe that BOP of Y_j implies BOP of Y by Theorem 1.2 (i), applied with P a singleton. Since Y is Stein, BOP implies that Y is elliptic (see Def. 2.2 below); for the simple proof see [13, Proposition 1.2] or [16, 3.2.A.]. By Theorem 2.4 below it follows that Y also enjoys POP. By part (iii) of Theorem 1.2, POP descends from $Y = Y_m$ to every Y_j. □

Remark 1.4. A main remaining open problem is whether the implication

$$\text{BOP} \Longrightarrow \text{POP} \tag{1.2}$$

holds for all complex manifolds. By using results of this paper and of his earlier works, F. Lárusson proved this implication for a large class of manifolds, including all quasi-projective manifolds [24, Theorem 4]. The main observation is that there exists an affine holomorphic fiber bundle $\pi\colon E \to \mathbb{P}^n$ with fiber $\mathbb{C}^n$ whose total space E is Stein; since the map π is acyclic and the fiber satisfies POP, the implication (1.2) follows from Corollary 1.3 (ii) for any closed complex subvariety $Y \subset \mathbb{P}^n$ (since the total space $E|_Y = \pi^{-1}(Y)$ is Stein). The same applies to complements of hypersurfaces in such Y; the higher codimension case reduces to the hypersurface case by blowing up. □

2. Subelliptic submersions and Serre fibrations

A holomorphic map $h\colon Z \to X$ of complex spaces is a *holomorphic submersion* if for every point $z_0 \in Z$ there exist an open neighborhood $V \subset Z$ of z_0, an open neighborhood $U \subset X$ of $x_0 = h(z_0)$, an open set W in a Euclidean space $\mathbb{C}^p$, and a biholomorphic map $\phi\colon V \to U \times W$ such that $pr_1 \circ \phi = h$, where $pr_1 : U \times W \to U$ is the projection on the first factor.

$$\begin{array}{ccc} Z \supset V & \xrightarrow{\ \phi\ } & U \times W \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle pr_1} \\ X \supset U & \xrightarrow{\ id\ } & U \end{array}$$

Each fiber $Z_x = h^{-1}(x)$ $(x \in X)$ of a holomorphic submersion is a closed complex submanifold of Z. A simple example is the restriction of a holomorphic fiber bundle projection $E \to X$ to an open subset Z of E.

We recall from [16, 4] the notion of a holomorphic spray and domination.

Definition 2.1. Assume that X and Z are complex spaces and $h\colon Z \to X$ is a holomorphic submersion. For $x \in X$ let $Z_x = h^{-1}(x)$.

(i) A *fiber-spray* on Z is a triple (E, π, s) consisting of a holomorphic vector bundle $\pi\colon E \to Z$ together with a holomorphic map $s\colon E \to Z$ such that for each $z \in Z$ we have
$$s(0_z) = z, \quad s(E_z) \subset Z_{h(z)}.$$

(ii) A spray (E, π, s) is *dominating* at a point $z \in Z$ if its differential
$$(\mathrm{d}s)_{0_z} : \mathrm{T}_{0_z}E \to \mathrm{T}_z Z$$
at the origin $0_z \in E_z = \pi^{-1}(z)$ maps the subspace E_z of $\mathrm{T}_{0_z}E$ surjectively onto the *vertical tangent space* $\mathrm{VT}_z Z = \ker \mathrm{d}h_z$. The spray is *dominating* (on Z) if it is dominating at every point $z \in Z$.

(iii) A family of h-sprays (E_j, π_j, s_j) $(j = 1, \ldots, m)$ on Z is dominating at the point $z \in Z$ if
$$(\mathrm{d}s_1)_{0_z}(E_{1,z}) + (\mathrm{d}s_2)_{0_z}(E_{2,z}) \cdots + (\mathrm{d}s_m)_{0_z}(E_{m,z}) = \mathrm{VT}_z Z.$$
If this holds for every $z \in Z$ then the family is *dominating* on Z.

(iv) A spray on a complex manifold Y is a fiber-spray associated to the constant map $Y \to$ *point*.

The simplest example of a spray on a complex manifold Y is the flow $Y \times \mathbb{C} \to Y$ of a $\mathbb{C}$-complete holomorphic vector field on Y. A composition of finitely many such flows, with independent time variables, is a dominating spray at every point where the given collection of vector fields span the tangent space of Y. Another example of a dominating spray is furnished by the exponential map on a complex Lie group G, translated over G by the group multiplication.

The following notion of an *elliptic submersion* is due to Gromov [16, Sect. 1.1.B]; *subelliptic submersions* were introduced in [4]. For examples see [4, 8, 16].

Definition 2.2. A holomorphic submersion $h\colon Z \to X$ is said to be *elliptic* (resp. *subelliptic*) if each point $x_0 \in X$ has an open neighborhood $U \subset X$ such that the restricted submersion $h\colon Z|_U \to U$ admits a dominating fiber-spray (resp. a finite dominating family of fiber-sprays). A complex manifold Y is elliptic (resp. subelliptic) if the trivial submersion $Y \to$ *point* is such.

The following notions appear in Theorem 1.2.

Definition 2.3. (a) A continuous map $\pi\colon E \to B$ is *Serre fibration* if it satisfies the homotopy lifting property for polyhedra (see [32, p. 8]).

(b) A holomorphic map $\pi\colon E \to B$ is an *elliptic Serre fibration* (resp. a *subelliptic Serre fibration*) if it is a surjective elliptic (resp. subelliptic) submersion and also a Serre fibration.

The following result was proved in [10] (see Theorems 1.4 and 8.3) by following the scheme proposed in [13, Sect. 7]. Earlier results include Gromov's Main Theorem [16, Theorem 4.5] (for elliptic submersions onto Stein manifolds, without interpolation), [13, Theorem 1.4] (for elliptic submersions onto Stein manifolds), [4, Theorem 1.1] (for subelliptic submersion), and [8, Theorem 1.2] (for fiber bundles with POP fibers over Stein manifolds).

Theorem 2.4. *Let $h\colon Z \to X$ be a holomorphic submersion of a complex space Z onto a Stein space X. Assume that X is exhausted by Stein Runge domains $D_1 \Subset D_2 \Subset \cdots \subset X = \bigcup_{j=1}^{\infty} D_j$ such that every D_j admits a stratification*

$$D_j = X_0 \supset X_1 \supset \cdots \supset X_{m_j} = \emptyset \tag{2.1}$$

with smooth strata $S_k = X_k \backslash X_{k+1}$ such that the restriction of $Z \to X$ to every connected component of each S_k is a subelliptic submersion, or a holomorphic fiber bundle with POP *fiber. Then sections $X \to Z$ satisfy* POP.

Remark 2.5. In previous papers [11, 12, 13, 4, 8, 9] POP was only considered for pairs of parameter spaces $P_0 \subset P$ such that

(*) P is a nonempty compact Hausdorff space, and P_0 is a closed subset of P that is a strong deformation neighborhood retract (SDNR) in P.

Here we dispense with the SDNR condition by using the Tietze extension theorem for maps into Hilbert spaces (see the proof of Proposition 4.4).

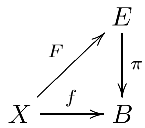
Theorem 2.4 also hold when P is a locally compact and countably compact Hausdorff space, and P_0 is a closed subspace of P. The proof requires only a minor change of the induction scheme (applying the diagonal process).

On the other hand, all stated results remain valid if we restrict to pairs $P_0 \subset P$ consisting of finite polyhedra; this suffices for most applications. □

Theorem 2.4 implies the following result concerning holomorphic liftings.

Theorem 2.6. *Let $\pi\colon E \to B$ be a holomorphic submersion of a complex space E onto a complex space B. Assume that B admits a stratification $B = B_0 \supset B_1 \supset \cdots \supset B_m = \emptyset$ by closed complex subvarieties such that the restriction of π to every connected component of each difference $B_j \backslash B_{j+1}$ is a subelliptic submersion, or a holomorphic fiber bundle with* POP *fiber.*

Given a Stein space X and a holomorphic map $f\colon X \to B$, every continuous lifting $F\colon X \to E$ of f $(\pi \circ F = f)$ is homotopic through liftings of f to a holomorphic lifting.



Proof. Assume first that X is finite dimensional. Then there is a stratification $X = X_0 \supset X_1 \supset \cdots \supset X_l = \emptyset$ by closed complex subvarieties, with smooth differences $S_j = X_j \backslash X_{j+1}$, such that each connected component S of every S_j is mapped by f to a stratum $B_k \backslash B_{k+1}$ for some $k = k(j)$. The pull-back submersion

$$f^*E = \{(x,e) \in X \times E\colon f(x) = \pi(e)\} \longmapsto X$$

then satisfies the assumptions of Theorem 2.4 with respect to this stratification of X. Note that liftings $X \to E$ of $f\colon X \to B$ correspond to sections $X \to f^*E$, and hence the result follows from Theorem 2.4. The general case follows by induction over an exhaustion of X by an increasing sequence of relatively compact Stein Runge domains in X. □

A fascinating application of Theorem 2.6 has recently been found by Ivarsson and Kutzschebauch [19, 20] who solved the following *Gromov's Vaserstein problem*:

Theorem 2.7. (Ivarsson and Kutzschebauch [19, 20]) *Let X be a finite-dimensional reduced Stein space and let $f\colon X \to \mathrm{SL}_m(\mathbb{C})$ be a null-homotopic holomorphic mapping. Then there exist a natural number N and holomorphic mappings $G_1, \ldots, G_N\colon X \to \mathbb{C}^{m(m-1)/2}$ (thought of as lower resp. upper triangular matrices) such that*

$$f(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix}$$

is a product of upper and lower diagonal unipotent matrices. (For odd N the last matrix has $G_N(x)$ in the lower left corner.)

In this application one takes $B = SL_m(\mathbb{C})$, $E = (\mathbb{C}^{m(m-1)/2})^N$, and $\pi\colon E \to B$ is the map

$$\pi(G_1, G_2, \ldots, G_N) = \begin{pmatrix} 1 & 0 \\ G_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_N \\ 0 & 1 \end{pmatrix}.$$

Every null-homotopic holomorphic map $f\colon X \to B = SL_m(\mathbb{C})$ admits a continuous lifting $F\colon X \to E$ for a suitably chosen $N \in \mathbb{N}$ (Vaserstein [31]), and the goal is to deform F to a holomorphic lifting $G = (G_1, \ldots, G_N)\colon X \to E$. This is done inductively by applying Theorem 2.6 to auxiliary submersions obtained by composing π with certain row projections. Stratified elliptic submersions naturally appear in their proof.

3. Convex approximation property

In this section we recall from [8] a characterization of Oka properties in terms of an Oka-Weil approximation property for entire maps $\mathbb{C}^n \to Y$.

Let $z = (z_1, \ldots, z_n)$, $z_j = x_j + \mathrm{i}\, y_j$, be complex coordinates on $\mathbb{C}^n$. Given numbers $a_j, b_j > 0$ $(j = 1, \ldots, n)$ we set

$$Q = \{z \in \mathbb{C}^n \colon |x_j| \le a_j,\ |y_j| \le b_j,\ j = 1, \ldots, n\}. \tag{3.1}$$

Definition 3.1. A *special convex set* in $\mathbb{C}^n$ is a compact convex set of the form

$$K = \{z \in Q \colon y_n \le \phi(z_1, \ldots, z_{n-1}, x_n)\}, \tag{3.2}$$

where Q is a cube (3.1) and ϕ is a continuous concave function with values in $(-b_n, b_n)$. Such (K, Q) is called a *special convex pair* in $\mathbb{C}^n$.

Definition 3.2. A complex manifold Y enjoys the *Convex Approximation Property* (CAP) if every holomorphic map $f \colon K \to Y$ on a special convex set $K \subset Q \subset \mathbb{C}^n$ (3.2) can be approximated, uniformly on K, by holomorphic maps $Q \to Y$.

Y enjoys the *Parametric Convex Approximation Property* (PCAP) if for every special convex pair (K, Q) and for every pair of parameter spaces $P_0 \subset P$ as in Def. 1.1, a map $f \colon Q \times P \to Y$ such that $f_p = f(\cdot\,, p) \colon Q \to Y$ is holomorphic for every $p \in P_0$, and is holomorphic on K for every $p \in P$, can be approximated uniformly on $K \times P$ by maps $\widetilde{f} \colon Q \times P \to Y$ such that $\widetilde{f}_p$ is holomorphic on Q for all $p \in P$, and $\widetilde{f}_p = f_p$ for all $p \in P_0$.

The following characterization of the Oka property was found in [8, 9] (for Stein source manifolds), thereby answering a question of Gromov [16, p. 881, 3.4.(D)]. For the extension to Stein source spaces see [10].

Theorem 3.3. *For every complex manifold we have*

$$\mathrm{BOP} \iff \mathrm{CAP}, \qquad \mathrm{POP} \iff \mathrm{PCAP}.$$

Remark 3.4. The implication PCAP $\Longrightarrow$ POP also holds for a specific pair of (compact, Hausdorff) parameter spaces as is seen from the proof in [8]. More precisely, if a complex manifold Y enjoys PCAP for a certain pair $P_0 \subset P$, then it also satisfies POP for that same pair. □

4. A parametric Oka principle for liftings

In this section we prove the main result of this paper, Theorem 4.2, which generalizes Theorem 2.6 to families of holomorphic maps. We begin by recalling the relevant terminology from [13].

Definition 4.1. Let $h \colon Z \to X$ be a holomorphic map of complex spaces, and let $P_0 \subset P$ be topological spaces.

(a) A *P-section* of $h \colon Z \to X$ is a continuous map $f \colon X \times P \to Z$ such that $f_p = f(\cdot\,, p) \colon X \to Z$ is a section of h for each $p \in P$. Such f is *holomorphic* if f_p is holomorphic on X for each fixed $p \in P$. If K is a compact set in X and if X' is a closed complex subvariety of X, then f is *holomorphic on* $K \cup X'$ if there is an open set $U \subset X$ containing K such that the restrictions $f_p|_U$ and $f_p|_{X'}$ are holomorphic for every $p \in P$.

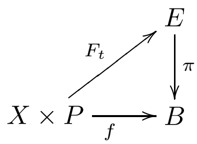
(b) A *homotopy of P-sections* is a continuous map $H\colon X \times P \times [0,1] \to Z$ such that $H_t = H(\cdot\,,\cdot\,,t)\colon X \times P \to Z$ is a P-section for each $t \in [0,1]$.

(c) A *(P, P_0)-section* of h is a P-section $f\colon X \times P \to Z$ such that $f_p = f(\cdot\,,p)\colon X \to Z$ is holomorphic on X for each $p \in P_0$. A (P, P_0)-section is holomorphic on a subset $U \subset X$ if $f_p|_U$ is holomorphic for every $p \in P$.

(d) A P-map $X \to Y$ to a complex space Y is a map $X \times P \to Y$. Similarly one defines (P, P_0)-maps and their homotopies.

Theorem 4.2. *Assume that E and B are complex spaces and $\pi\colon E \to B$ is a subelliptic submersion* (*Def.* 2.3), *or a holomorphic fiber bundle with* POP *fiber* (*Def.* 1.1). *Let P be a compact Hausdorff space and P_0 a closed subspace of P. Given a Stein space X, a compact $\mathcal{O}(X)$-convex subset K of X, a closed complex subvariety X' of X, a holomorphic P-map $f\colon X \times P \to B$, and a (P, P_0)-map $F\colon X \times P \to E$ that is a π-lifting of f ($\pi \circ F = f$) and is holomorphic on* (*a neighborhood of*) *K and on the subvariety X', there exists a homotopy of liftings $F^t\colon X \times P \to E$ of f ($t \in [0,1]$) that is fixed on $(X \times P_0) \cup (X' \times P)$, that approximates $F = F^0$ uniformly on $K \times P$, and such that F^1_p is holomorphic on X for all $p \in P$.*

If in addition F is holomorphic in a neighborhood of $K \cup X'$ then the homotopy F^t can be chosen such that it agrees with F^0 to a given finite order along X'.



Definition 4.3. A map $\pi\colon E \to B$ satisfying the conclusion of Theorem 4.2 is said to enjoy the parametric Oka property (c.f. Lárusson [22, 23, 24]). □

Proof. The first step is a reduction to the graph case. Set $Z = X \times E$, $\widetilde{Z} = X \times B$, and let $\widetilde{\pi}\colon Z \to \widetilde{Z}$ denote the map

$$\widetilde{\pi}(x, e) = (x, \pi(e)), \quad x \in X,\ e \in E.$$

Then $\widetilde{\pi}$ is a subelliptic submersion, resp. a holomorphic fiber bundle with POP fiber. Let $\widetilde{h}\colon \widetilde{Z} = X \times B \to X$ denote the projection onto the first factor, and let $h = \widetilde{h} \circ \widetilde{\pi}\colon Z \to X$. To a P-map $f\colon X \times P \to B$ we associate the P-section $\widetilde{f}(x,p) = (x, f(x,p))$ of $\widetilde{h}\colon \widetilde{Z} \to X$. Further, to a lifting $F\colon X \times P \to E$ of f we associate the P-section $\widetilde{F}(x,p) = (x, F(x,p))$ of $h\colon Z \to X$. Then $\widetilde{\pi} \circ \widetilde{F} = \widetilde{f}$. This allows us to drop the tilde's on π, f and F and consider from now on the following situation:

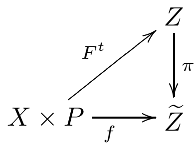
(i) Z and $\widetilde{Z}$ are complex spaces,

(ii) $\pi\colon Z \to \widetilde{Z}$ is a subelliptic submersion, or a holomorphic fiber bundle with POP fiber,

(iii) $\widetilde{h}\colon \widetilde{Z} \to X$ is a holomorphic map onto a Stein space X,
(iv) $f\colon X \times P \to \widetilde{Z}$ is a holomorphic P-section of $\widetilde{h}$,
(v) $F\colon X \times P \to Z$ is a holomorphic (P, P_0)-section of $h = \widetilde{h} \circ \pi\colon Z \to X$ such that $\pi \circ F = f$, and F is holomorphic on $K \cup X'$.

We need to find a homotopy $F^t\colon X \times P \to Z$ $(t \in [0,1])$ consisting of (P, P_0)-sections of $h\colon Z \to X$ such that $\pi \circ F^t = f$ for all $t \in [0,1]$, and

(α) $F^0 = F$,
(β) F^1 is a holomorphic P-section, and
(γ) for every $t \in [0,1]$, F^t is holomorphic on K, it is uniformly close to F^0 on $K \times P$, and it agrees with F^0 on $(X \times P_0) \cup (X' \times P)$.



Set $f_p = f(\cdot\,, p)\colon X \to \widetilde{Z}$ for $p \in P$. The image $f_p(X)$ is a closed Stein subspace of $\widetilde{Z}$ that is biholomorphic to X (since $\widetilde{h} \circ f$ is the identity on X).

When $P = \{p\}$ is a singleton, there is only one section $f = f_p$, and the desired conclusion follows by applying Theorem 2.4 to the restricted submersion $\pi\colon Z|_{f(X)} \to f(X)$.

In general we consider the family of restricted submersions $Z|_{f_p(X)} \to f_p(X)$ $(p \in P)$. The proof of the parametric Oka principle [12, Theorem 1.4] requires certain modifications that we now explain. It suffices to obtain a homotopy F^t of liftings of f over a relatively compact subset D of X with $K \subset D$; the proof is then finished by induction over an exhaustion of X. The initial step is provided by the following proposition. (No special assumption is needed on the submersion $\pi\colon Z \to \widetilde{Z}$ for this result.)

Proposition 4.4. (Assumptions as above) *Let D be an open relatively compact set in X with $K \subset D$. There exists a homotopy of liftings of f over D from $F = F^0|_{D \times P}$ to a lifting F' such that properties* (α) *and* (γ) *hold for F', while* (β) *is replaced by*

(β') *F'_p is holomorphic on D for all p in a neighborhood $P'_0 \subset P$ of P_0.*

The existence of such local holomorphic extension F' is used at several subsequent steps. We postpone the proof of the proposition to the end of this section and continue with the proof of Theorem 4.2. Replacing F by F' and X by D, we assume from now on that F_p is holomorphic on X for all $p \in P'_0$ (a neighborhood of P_0).

Assume for the sake of discussion that X is a Stein manifold, that $X' = \emptyset$, and that $\pi\colon Z \to \widetilde{Z}$ is a subelliptic submersion. (The proof in the fiber bundle case is simpler and will be indicated along the way. The case when X has singularities or $X' \neq \emptyset$ uses the induction scheme from [10], but the details presented here remain unchanged.) It suffices to explain the following:

Main step: Let $K \subset L$ be compact strongly pseudoconvex domains in X that are $\mathcal{O}(X)$-convex. Assume that $F^0 = \{F^0_p\}_{p\in P}$ is a π-lifting of $f = \{f_p\}_{p\in P}$ such that F^0_p is holomorphic on K for all $p \in P$, and F^0_p is holomorphic on X when $p \in P'_0$. Find a homotopy of liftings $F^t = \{F^t_p\}_{p\in P}$ ($t \in [0,1]$) that are holomorphic on K, uniformly close to F^0 on $K \times P$, the homotopy is fixed for all p in a neighborhood of P_0, and F^1_p is holomorphic on L for all $p \in P$.

Granted the Main Step, a solution satisfying the conclusion of Theorem 4.2 is then obtained by induction over a suitable exhaustion of X.

Proof of the Main Step. We cover the compact set $\bigcup_{p\in P} f_p(\overline{L\backslash K}) \subset \widetilde{Z}$ by open sets $U_1, \ldots, U_N \subset \widetilde{Z}$ such that every restricted submersion $\pi\colon Z|_{U_j} \to U_j$ admits a finite dominating family of π-sprays. In the fiber bundle case we choose the sets U_j such that $Z|_{U_j}$ is isomorphic to the trivial bundle $U_j \times Y \to U_j$ with POP fiber Y.

Choose a Cartan string $\mathcal{A} = (A_0, A_1, \ldots, A_n)$ in X [12, Def. 4.2] such that $K = A_0$ and $L = \bigcup_{j=0}^n A_j$. The construction is explained in [12, Corollary 4.5]: It suffices to choose each of the compact sets A_k to be a strongly pseudoconvex domain such that $\left(\bigcup_{j=0}^{k-1} A_j, A_k\right)$ is a Cartan pair for all $k = 1, \ldots, n$. In addition, we choose the sets $A_1, \ldots, A_n$ small enough such that $f_p(A_j)$ is contained in one of the sets U_l for every $p \in P$ and $j = 1, \ldots, n$.

We cover P by compact subsets $P_1, \ldots, P_m$ such that for every $j = 1, \ldots, m$ and $k = 1, \ldots, n$, there is a neighborhood $P'_j \subset P$ of P_j such that the set $\bigcup_{p\in P'_j} f_p(A_k)$ is contained in one of the sets U_l.

As in [12] we denote by $\mathcal{K}(\mathcal{A})$ the *nerve complex* of $\mathcal{A} = (A_0, A_1, \ldots, A_n)$, i.e., a combinatorial simplicial complex consisting of all multiindices $J = (j_0, j_1, \ldots, j_k)$, with $0 \le j_0 < j_1 < \cdots < j_k \le n$, such that

$$A_J = A_{j_0} \cap A_{j_1} \cap \cdots \cap A_{j_k} \neq \emptyset.$$

Its *geometric realization*, $K(\mathcal{A})$, is a finite polyhedron in which every multiindex $J = (j_0, j_1, \ldots, j_k) \in \mathcal{K}(\mathcal{A})$ of length $k+1$ determines a closed k-dimensional face $|J| \subset K(\mathcal{A})$, homeomorphic to the standard k-simplex in $\mathbb{R}^k$, and every k-dimensional face of $K(\mathcal{A})$ is of this form. The face $|J|$ is called the *body* (or *carrier*) of J, and J is the *vertex scheme* of $|J|$. Given $I, J \in \mathcal{K}(\mathcal{A})$ we have $|I| \cap |J| = |I \cup J|$. The vertices of $K(\mathcal{A})$ correspond to the individual sets A_j in $\mathcal{A}$, i.e., to singletons $(j) \in \mathcal{K}(\mathcal{A})$. (See [18] or [30] for simplicial complexes and polyhedra.)

Given a compact set A in X, we denote by $\Gamma_{\mathcal{O}}(A, Z)$ the space of all sections of $h\colon Z \to X$ that are holomorphic over some unspecified open neighborhood A in Z, in the sense of germs at A.

Recall that a *holomorphic* $\mathcal{K}(\mathcal{A}, Z)$*-complex* [12, Def. 3.2] is a continuous family of holomorphic sections

$$F_* = \{F_J\colon |J| \to \Gamma_{\mathcal{O}}(A_J, Z), \quad J \in \mathcal{K}(\mathcal{A})\}$$

satisfying the following compatibility conditions:

$$I, J \in \mathcal{K}(\mathcal{A}),\ I \subset J \Longrightarrow F_J(t) = F_I(t)|_{A_J} \ (\forall t \in |I|).$$

Note that

- $F_{(k)}$ a holomorphic section over (a neighborhood of) A_k,
- $F_{(k_0,k_1)}$ is a homotopy of holomorphic sections over $A_{k_0} \cap A_{k_1}$ connecting $F_{(k_0)}$ and $F_{(k_1)}$,
- $F_{(k_0,k_1,k_2)}$ is a triangle of homotopies with vertices $F_{(k_0)}, F_{(k_1)}, F_{(k_2)}$ and sides $F_{(k_0,k_1)}, F_{(k_0,k_2)}, F_{(k_1,k_2)}$, etc.

Similarly one defines a continuous $\mathcal{K}(\mathcal{A}, Z)$-complex.

A $\mathcal{K}(\mathcal{A}, Z; P)$-complex is defined in an obvious way by adding the parameter $p \in P$. It can be viewed as a $\mathcal{K}(\mathcal{A}, Z)$-complex of P-sections of $Z \to X$, or as a family of $\mathcal{K}(\mathcal{A}, Z)$-complexes depending continuously on the parameter $p \in P$. Similarly, a $\mathcal{K}(\mathcal{A}, Z; P, P_0)$-complex is a $\mathcal{K}(\mathcal{A}, Z; P)$-complex consisting of holomorphic sections (over the set $L = \bigcup_{j=0}^{n} A_j$) for the parameter values $p \in P_0$. The terminology of Def. 4.1 naturally applies to complexes of sections.

By choosing the sets $A_1, \ldots, A_n$ sufficiently small and by shrinking the neighborhood P_0' (furnished by Proposition 4.4) around P_0 if necessary we can deform $F = F^0$ to a holomorphic $\mathcal{K}(\mathcal{A}, Z; P, P_0')$-complex $F_{*,*} = \{F_{*,p}\}_{p \in P}$ such that

- every section in $F_{*,p}$ projects by $\pi\colon Z \to \widetilde{Z}$ to the section f_p (such $F_{*,*}$ is called a *lifting* of the holomorphic P-section $f = \{f_p\}_{p \in P}$),
- $F_{(0),p}$ is the restriction to $A_0 = K$ of the initial section F_p^0, and
- for $p \in P_0'$, every section in $F_{*,p}$ is the restriction of F_p^0 to the appropriate subdomain (i.e., the deformation from F^0 to $F_{*,*}$ is fixed over P_0').

A completely elementary construction of such *initial holomorphic complex* $F_{*,*}$ can be found in [12, Proposition 4.7].

Remark 4.5. We observe that, although the map $h = \widetilde{h} \circ \pi\colon Z \to X$ is not necessarily a submersion (since the projection $\widetilde{h}\colon \widetilde{Z} \to X$ may have singular fibers), the construction in [12] still applies since we only work with the fiber component of F_p (over f_p) with respect to the submersion $\pi\colon Z \to \widetilde{Z}$. All lifting problems locally reduce to working with functions. □

The rest of the construction amounts to finitely many homotopic modifications of the complex $F_{*,*}$. At every step we collapse one of the cells in the complex and obtain a family (parametrized by P) of holomorphic sections over the union of the sets that determine the cell. In finitely many steps we obtain a family of *constant complexes* $F^1 = \{F_p^1\}_{p \in P}$, that is, F_p^1 is a holomorphic section of $Z \to X$ over L. This procedure is explained in [12, Sect. 5] (see in particular Proposition 5.1.). The additional lifting condition is easily satisfied at every step of the construction. In the end, the homotopy of complexes from F^0 to F^1 is replaced by a homotopy of constant complexes, i.e., a homotopy of liftings F^t of f that consist of sections over L (see the conclusion of proof of Theorem 1.5 in [12, p. 657]).

Let us describe more carefully the main step – collapsing a segment in a holomorphic complex. (All substeps in collapsing a cell reduce to collapsing a segment, each time with an additional parameter set.)

We have a special pair (A, B) of compact sets contained in $L \subset X$, called a *Cartan pair* [13, Def. 4], with B contained in one of the sets $A_1, \dots, A_n$ in our Cartan string $\mathcal{A}$. (Indeed, B is the intersection of some of these sets.) Further, we have an additional compact parameter set $\widetilde{P}$ (which appears in the proof) and families of holomorphic sections of $h\colon Z \to X$, $a_{(p,\tilde{p})}$ over A and $b_{(p,\tilde{p})}$ over B, depending continuously on $(p, \tilde{p}) \in P \times \widetilde{P}$ and projecting by $\pi\colon Z \to \widetilde{Z}$ to the section f_p. For $p \in P_0'$ we have $a_{(p,\tilde{p})} = b_{(p,\tilde{p})}$ over $A \cap B$. These two families are connected over $A \cap B$ by a homotopy of holomorphic sections $b^t_{(p,\tilde{p})}$ ($t \in [0, 1]$) such that

$$b^0_{(p,\tilde{p})} = a_{(p,\tilde{p})}, \quad b^1_{(p,\tilde{p})} = b_{(p,\tilde{p})}, \quad \pi \circ b^t_{(p,\tilde{p})} = f_p$$

hold for each $p \in P$ and $t \in [0, 1]$, and the homotopy is fixed for $p \in P_0'$. These two families are joined into a family of holomorphic sections $\widetilde{a}_{(p,\tilde{p})}$ over $A \cup B$, projecting by π to f_p. The deformation consists of two substeps:

1. by applying the Oka-Weil theorem [11, Theorem 4.2] over the pair $A \cap B \subset B$ we approximate the family $a_{(p,\tilde{p})}$ sufficiently closely, uniformly on a neighborhood of $A \cap B$, by a family $\widetilde{b}_{(p,\tilde{p})}$ of holomorphic sections over B;
2. assuming that the approximation in (1) is sufficiently close, we glue the families $a_{(p,\tilde{p})}$ and $\widetilde{b}_{(p,\tilde{p})}$ into a family of holomorphic sections $\widetilde{a}_{(p,\tilde{p})}$ over $A \cup B$ such that $\pi \circ \widetilde{a}_{(p,\tilde{p})} = f_p$.

For Substep (2) we can use local holomorphic sprays as in [8, Proposition 3.1], or we apply [11, Theorem 5.5]. The projection condition $\pi \circ \widetilde{a}_{(p,\tilde{p})} = f_p$ is a trivial addition.

Substep (1) is somewhat more problematic as it requires a dominating family of π-sprays on $Z|_U$ over an open set $U \subset \widetilde{Z}$ to which the sections $b^t_{(p,\tilde{p})}$ project. (In the fiber bundle case we need triviality of the restricted bundle $Z|_U \to U$ and POP of the fiber.) Recall that B is contained in one of the sets A_k, and therefore

$$\bigcup_{p \in P_j'} f_p(B) \subset \bigcup_{p \in P_j'} f_p(A_k) \subset U_{l(j,k)}.$$

Since $\pi \circ b^t_{(p,\tilde{p})} = f_p$ and Z admits a dominating family of π-sprays over each set U_l, Substep (1) applies separately to each of the m families

$$\{b^t_{(p,\tilde{p})} \colon p \in P_j',\ \tilde{p} \in \widetilde{P},\ t \in [0, 1]\}, \quad j = 1, \dots, m.$$

To conclude the proof of the Main Step we use the *stepwise extension method*, similar to the one in [12, pp. 138–139]. In each step we make the lifting holomorphic for the parameter values in one of the sets P_j, keeping the homotopy fixed over the union of the previous sets.

We begin with P_1. The above shows that the Main Step can be accomplished in finitely many applications of Substeps (1) and (2), using the pair of parameter spaces $P_0 \cap P_1' \subset P_1'$ (instead of $P_0 \subset P$). We obtain a homotopy of liftings

$\{F_p^t \colon p \in P_1', t \in [0,1]\}$ of f_p such that F_p^1 is holomorphic on L for all p in a neighborhood of P_1, and $F_p^t = F_p^0$ for all $t \in [0,1]$ and all p in a relative neighborhood of $P_0 \cap P_1'$ in P_1'. We extend this homotopy to all values $p \in P$ by replacing F_p^t by $F_p^{t\chi(p)}$, where $\chi \colon P \to [0,1]$ is a continuous function that equals one near P_1 and has support contained in P_1'. Thus F_p^1 is holomorphic on L for all p in a neighborhood V_1 of $P_0 \cup P_1$, and $F_p^1 = F_p^0$ for all p in a neighborhood of P_0.

We now repeat the same procedure with F^1 as the 'initial' lifting of f, using the pair of parameter spaces $(P_0 \cup P_1) \cap P_2' \subset P_2'$. We obtain a homotopy of liftings $\{F_p^t\}_{t\in[1,2]}$ of f_p for $p \in P_2'$ such that the homotopy is fixed for all p in a neighborhood of $(P_0 \cup P_1) \cap P_2'$ in P_2', and F_p^2 is holomorphic on L for all p in a neighborhood of $P_0 \cup P_1 \cup P_2$ in P.

In m steps of this kind we get a homotopy $\{F^t\}_{t\in[0,m]}$ of liftings of f such that F_p^m is holomorphic on L for all $p \in P$, and the homotopy is fixed in a neighborhood of P_0 in P. It remains to rescale the parameter interval $[0,m]$ back to $[0,1]$.

This concludes the proof in the special case when X is a Stein manifold and $X' = \emptyset$. In the general case we follow the induction scheme in the proof of the parametric Oka principle for stratified fiber bundles with POP fibers in [10]; Cartan strings are now used inside the smooth strata.

When $\pi \colon Z \to \widetilde{Z}$ is a fiber bundle, we apply the one-step approximation and gluing procedure as in [8], without having to deal with holomorphic complexes. The Oka-Weil approximation theorem in Substep (1) is replaced by POP of the fiber. $\square$

Proof of Proposition 4.4. We begin by considering the special case when $\pi \colon Z = \widetilde{Z} \times \mathbb{C} \to \widetilde{Z}$ is a trivial line bundle. We have $F_p = (f_p, g_p)$ where g_p is a holomorphic function on X for $p \in P_0$, and is holomorphic on $K \cup X'$ for all $p \in P$. We replace X by a relatively compact subset containing $\bar{D}$ and consider it as a closed complex subvariety of a Euclidean space $\mathbb{C}^N$. Choose bounded pseudoconvex domains $\Omega \Subset \Omega'$ in $\mathbb{C}^N$ such that $\bar{D} \subset \Omega \cap X$.

By [13, Lemma 3.1] there exist bounded linear extension operators

$$\begin{aligned} S \colon H^\infty(X \cap \Omega') &\longrightarrow H^2(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega), \\ S' \colon H^\infty(X' \cap \Omega') &\longrightarrow H^2(\Omega), \end{aligned}$$

such that $S(g)|_{X\cap\Omega} = g|_{X\cap\Omega}$, and likewise for S'. (In [13] we obtained an extension operator into $H^\infty(\Omega)$, but the Bergman space appeared as an intermediate step. Unlike the Ohsawa-Takegoshi extension theorem [26], this is a soft result depending on the Cartan extension theorem and some functional analysis; the price is shrinking of the domain.) Set

$$h_p = S(g_p|_{X\cap\Omega'}) - S'(g_p|_{X'\cap\Omega'}) \in H^2(\Omega), \qquad p \in P_0.$$

Then h_p vanishes on X', and hence it belongs to the closed subspace $H^2_{X'}(\Omega)$ consisting of all functions in $H^2(\Omega)$ that vanish on $X' \cap \Omega$. Since these are Hilbert spaces, the generalized Tietze extension theorem (a special case of Michael's convex selection theorem; see [28, Part C, Theorem 1.2, p. 232] or [3, 25]) furnishes a continuous extension of the map $P_0 \to H^2_{X'}(\Omega)$, $p \to h_p$, to a map $P \ni p \to \widetilde{h}_p \in H^2_{X'}(\Omega)$. Set

$$G_p = \widetilde{h}_p + S'(g_p|_{X' \cap \Omega'}) \in H^2(\Omega), \qquad p \in P.$$

Then

$$G_p|_{X' \cap \Omega} = g_p|_{X' \cap \Omega} \; (\forall p \in P), \quad G_p|_{X \cap \Omega} = g_p|_{X \cap \Omega} \; (\forall p \in P_0).$$

This solves the problem, except that G_p should approximate g_p uniformly on K. Choose holomorphic functions $\phi_1, \ldots, \phi_m$ on $\mathbb{C}^N$ that generate the ideal sheaf of the subvariety X' at every point in Ω'. A standard application of Cartan's Theorem B shows that in a neighborhood of K we have

$$g_p = G_p + \sum_{j=1}^{m} \phi_j \, \xi_{j,p}$$

for some holomorphic functions $\xi_{j,p}$ in a neighborhood of K, depending continuously on $p \in P$ and vanishing identically on X for $p \in P_0$. (See, e.g., [12, Lemma 8.1].)

Since the set K is $\mathcal{O}(X)$-convex, and hence polynomially convex in $\mathbb{C}^N$, an extension of the Oka-Weil approximation theorem (using a bounded linear solution operator for the $\overline{\partial}$-equation, given for instance by Hörmander's L^2-methods [17] or by integral kernels) furnishes functions $\widetilde{\xi}_{j,p} \in \mathcal{O}(\Omega)$, depending continuously on $p \in P$, such that $\widetilde{\xi}_{j,p}$ approximates $\xi_{j,p}$ as close as desired uniformly on K, and it vanishes on $X \cap \Omega$ when $p \in P_0$. Setting

$$\widetilde{g}_p = G_p + \sum_{j=1}^{m} \phi_j \, \widetilde{\xi}_{j,p}, \quad p \in P$$

gives the solution. This proof also applies to vector-valued maps by applying it componentwise.

The general case reduces to the special case by using that for every $p_0 \in P_0$, the Stein subspace $F_{p_0}(X)$ (resp. $f_{p_0}(X)$) admits an open Stein neighborhood in Z (resp. in $\widetilde{Z}$) according to a theorem of Siu [2, 29]. Embedding these neighborhoods in Euclidean spaces and using holomorphic retractions onto fibers of π (see [10, Proposition 3.2]), the special case furnishes neighborhoods $U_{p_0} \subset U'_{p_0}$ of p_0 in P and a P-section $F' \colon \bar{D} \times P \to Z$, homotopic to F through liftings of f, such that

(i) $\pi \circ F'_p = f_p$ for all $p \in P$,
(ii) F'_p is holomorphic on $\bar{D}$ when $p \in U_{p_0}$,
(iii) $F'_p = F_p$ for $p \in P_0 \cup (P \backslash U'_{p_0})$,
(iv) $F'_p|_{X' \cap D} = F_p|_{X' \cap D}$ for all $p \in P$, and
(v) F' approximates F on $K \times P$.

(The special case is first used for parameter values p in a neighborhood U'_{p_0} of p_0; the resulting family of holomorphic maps $\bar{D} \times U'_{p_0} \to \mathbb{C}^N$ is then patched with F by using a cut-off function $\chi(p)$ with support in U'_{p_0} that equals one on a neighborhood U_{p_0} of p_0, and applying holomorphic retractions onto the fibers of π.) In finitely many steps of this kind we complete the proof. □

Remark 4.6. One might wish to extend Theorem 4.2 to the case when $\pi\colon E \to B$ is a *stratified* subelliptic submersion, or a stratified fiber bundle with POP fibers. The problem is that the induced stratifications on the pull-back submersions $f_p^* E \to X$ may change discontinuously with respect to the parameter p. Perhaps one could get a positive result by assuming that the stratification of $E \to B$ is suitably compatible with the variable map $f_p\colon X \to B$. □

Recall (Def. 4.3) that a holomorphic map $\pi\colon E \to B$ satisfies POP if the conclusion of Theorem 4.2 holds. We show that this is a local property.

Theorem 4.7. (Localization principle for POP) *A holomorphic submersion $\pi\colon E \to B$ of a complex space E onto a complex space B satisfies* POP *if and only if every point $x \in B$ admits an open neighborhood $U_x \subset B$ such that the restricted submersion $\pi\colon E|_{U_x} \to U_x$ satisfies* POP.

Proof. If $\pi\colon E \to B$ satisfies POP then clearly so does its restriction to any open subset U of B.

Conversely, assume that B admits an open covering $\mathcal{U} = \{U_\alpha\}$ by open sets such that every restriction $E|_{U_\alpha} \to U_\alpha$ enjoys POP. When proving POP for $\pi\colon E \to B$, a typical step amounts to choosing small compact sets $A_1, \ldots, A_n$ in the source (Stein) space X such that, for a given compact set $A_0 \subset X$, $\mathcal{A} = (A_0, A_1, \ldots, A_n)$ is a Cartan string. We can choose the sets $A_1, \ldots, A_n$ sufficiently small such that each map $f_p\colon X \to B$ in the given family sends each A_j into one of the sets $U_\alpha \in \mathcal{U}$.

To the string $\mathcal{A}$ we associate a $\mathcal{K}(\mathcal{A}, Z; P, P_0)$-complex $F_{*,*}$ which is then inductively deformed into a holomorphic P-map $\widetilde{F}\colon \bigcup_{j=0}^n A_j \times P \to E$ such that $\pi \circ \widetilde{F} = f$. The main step in the inductive procedure amounts to patching a pair of liftings over a Cartan pair (A', B') in X, where the set B' is contained in one of the sets $A_1, \ldots, A_n$ in the Cartan string $\mathcal{A}$. This is subdivided into substeps (1) and (2) (see the proof of Theorem 4.2). Only the first of these substeps, which requires a Runge-type approximation property, is a nontrivial condition on the submersion $E \to B$. It is immediate from the definitions that this approximation property holds if there is an open set $U \subset B$ containing the image $f_p(B')$ (for a certain set of parameter values $p \in P$) such that the restricted submersion $E|_U \to U$ satisfies POP. In our case this is so since we have insured that $f_p(B') \subset f_p(A_j) \subset U_\alpha$ for some $j \in \{1, \ldots, n\}$ and $U_\alpha \in \mathcal{U}$. □

5. Ascent and descent of the parametric Oka property

In this section we prove Theorem 1.2 stated in Section 1.

Proof of (i): Assume that B enjoys POP (which is equivalent to PCAP). Let (K, Q) be a special convex pair in $\mathbb{C}^n$ (Def. 3.1), and let $F\colon Q \times P \to E$ be a (P, P_0)-map that is holomorphic on K (Def. 4.1).

Then $f = \pi \circ F\colon Q \times P \to B$ is a (P, P_0)-map that is holomorphic on K. Since B enjoys POP, there is a holomorphic P-map $g\colon Q \times P \to B$ that agrees with f on $Q \times P_0$ and is uniformly close to f on a neighborhood of $K \times P$ in $\mathbb{C}^n \times P$.

If the latter approximation is close enough, there exists a holomorphic P-map $G\colon K \times P \to E$ such that $\pi \circ G = g$, G approximates F on $K \times P$, and $G = F$ on $K \times P_0$. To find such lifting of g, we consider graphs of these maps (as in the proof of Theorem 4.2) and apply a holomorphic retraction onto the fibers of π [10, Proposition 3.2].

Since $G = F$ on $K \times P_0$, we can extend G to $(K \times P) \cup (Q \times P_0)$ by setting $G = F$ on $Q \times P_0$.

Since $\pi\colon E \to B$ is a Serre fibration and K is a strong deformation retract of Q (these sets are convex), G extends to a continuous (P, P_0)-map $G\colon Q \times P \to E$ such that $\pi \circ G = g$. The extended map remains holomorphic on K.

By Theorem 4.2 there is a homotopy of liftings $G^t\colon Q \times P \to E$ of g $(t \in [0, 1])$ which is fixed on $Q \times P_0$ and is holomorphic and uniformly close to $G^0 = G$ on $K \times P$. The holomorphic P-map $G^1\colon Q \times P \to E$ then satisfies the condition in Def. 3.2 relative to F. This proves that E enjoys PCAP and hence POP.

Proof of (ii): Assume that E enjoys POP. Let (K, Q) be a special convex pair, and let $f\colon Q \times P \to B$ be a (P, P_0)-map that is holomorphic on K. Assuming that P is contractible, the Serre fibration property of $\pi\colon E \to B$ insures the existence of a continuous P-map $F\colon Q \times P \to E$ such that $\pi \circ F = f$. (The subset P_0 of P does not play any role here.) Theorem 4.2 furnishes a homotopy $F^t\colon Q \times P \to E$ $(t \in [0, 1])$ such that

(a) $F^0 = F$,
(b) $\pi \circ F^t = f$ for each $t \in [0, 1]$, and
(c) F^1 is a (P, P_0)-map that is holomorphic on K.

This is accomplished in two steps: We initially apply Theorem 4.2 with $Q \times P_0$ to obtain a homotopy $F^t\colon Q \times P_0 \to E$ $(t \in [0, \frac{1}{2}])$, satisfying properties (a) and (b) above, such that $F_p^{1/2}$ is holomorphic on Q for all $p \in P_0$. For trivial reasons this homotopy extends continuously to all values $p \in P$. In the second step we apply Theorem 4.2 over $K \times P$, with $F^{1/2}$ as the initial lifting of f and keeping the homotopy fixed for $p \in P_0$ (where it is already holomorphic), to get a homotopy F^t $(t \in [\frac{1}{2}, 1])$ such that $\pi \circ F^t = f$ and F_p^1 is holomorphic on K for all $p \in P$.

Since E enjoys POP, F^1 can be approximated uniformly on $K \times P$ by holomorphic P-maps

$$\widetilde{F}\colon Q \times P \to E$$

such that $\widetilde{F} = F^1$ on $Q \times P_0$.

Then

$$\widetilde{f} = \pi \circ \widetilde{F} \colon Q \times P \to B$$

is a holomorphic P-map that agrees with f on $Q \times P_0$ and is close to f on $K \times P$.

This shows that B enjoys PCAP for any contractible (compact, Hausdorff) parameter space P and for any closed subspace P_0 of P. Since the implication PCAP$\Longrightarrow$POP in Theorem 3.3 holds for each specific pair (P_0, P) of parameter spaces, we infer that B also enjoys POP for such parameter pairs. This completes the proof of (ii).

Proof of (iii): Contractibility of P was used in the proof of (ii) to lift the map $f \colon Q \times P \to B$ to a map $F \colon Q \times P \to E$. Such a lift exists for every topological space if $\pi \colon E \to B$ is a weak homotopy equivalence. This is because a Serre fibration between smooth manifolds is also a Hurewicz fibration (by Cauty [1]), and a weak homotopy equivalence between them is a homotopy equivalence by the Whitehead Lemma. □

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Added in proofs

Since the completion of this paper, the author gave a positive answer to the question posed in Remark 1.4 for parameter spaces $P_0 \subset P$ that are compact sets in a Euclidean space $\mathbb{R}^m$ (C. R. Acad. Sci. Paris, Ser. I **347**, 1017–1020 (2009); C. R. Acad. Sci. Paris, Ser. I (2009)).

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Positivity of the $\bar{\partial}$-Neumann Laplacian

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Dedicated to Professor Linda Rothschild

Abstract. We study the $\bar{\partial}$-Neumann Laplacian from spectral theoretic perspectives. In particular, we show how pseudoconvexity of a bounded domain is characterized by positivity of the $\bar{\partial}$-Neumann Laplacian.

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1. Introduction

Whether or not a given system has positive ground state energy is a widely studied problem with significant repercussions in physics, particularly in quantum mechanics. It follows from the classical Hardy inequality that the bottom of the spectrum of the Dirichlet Laplacian on a domain in $\mathbb{R}^n$ that satisfies the outer cone condition is positive if and only if its inradius is finite (see [D95]). Whereas spectral behavior of the Dirichlet Laplacian is insensitive to boundary geometry, the story for the $\bar{\partial}$-Neumann Laplacian is different. Since the work of Kohn [Ko63, Ko64] and Hörmander [H65], it has been known that existence and regularity of the $\bar{\partial}$-Neumann Laplacian closely depend on the underlying geometry (see the surveys [BSt99, Ch99, DK99, FS01] and the monographs [CS99, St09]).

Let Ω be a domain in $\mathbb{C}^n$. It follows from the classical Theorem B of Cartan that if Ω is pseudoconvex, then the Dolbeault cohomology groups $H^{0,q}(\Omega)$ vanish for all $q \geq 1$. (More generally, for any coherent analytic sheaf $\mathcal{F}$ over a Stein manifold, the sheaf cohomology groups $H^q(X, \mathcal{F})$ vanish for all $q \geq 1$.) The converse is also true ([Se53], p. 65). Cartan's Theorem B and its converse were generalized by Laufer [L66] and Siu [Siu67] to a Riemann domain over a Stein manifold. When Ω is bounded, it follows from Hörmander's L^2-existence theorem for the $\overline{\partial}$-operator

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that if Ω is in addition pseudoconvex, then the L^2-cohomology groups $\widetilde{H}^{0,q}(\Omega)$ vanish for $q \geq 1$. The converse of Hörmander's theorem also holds, under the assumption that the interior of the closure of Ω is the domain itself. Sheaf theoretic arguments for the Dolbeault cohomology groups can be modified to give a proof of this fact (cf. [Se53, L66, Siu67, Br83, O88]; see also [Fu05] and Section 3 below).

In this expository paper, we study positivity of the $\overline{\partial}$-Neumann Laplacian, in connection with the above-mentioned classical results, through the lens of spectral theory. Our emphasis is on the interplay between spectral behavior of the $\overline{\partial}$-Neumann Laplacian and the geometry of the domains. This is evidently motivated by Marc Kac's famous question "Can one hear the shape of a drum?" [Ka66]. Here we are interested in determining the geometry of a domain in $\mathbb{C}^n$ from the spectrum of the $\overline{\partial}$-Neumann Laplacian. (See [Fu05, Fu08] for related results.) We make an effort to present a more accessible and self-contained treatment, using extensively spectral theoretic language but bypassing sheaf cohomology theory.

2. Preliminaries

In this section, we review the spectral theoretic setup for the $\bar{\partial}$-Neumann Laplacian. The emphasis here is slightly different from the one in the extant literature (cf. [FK72, CS99]). The $\overline{\partial}$-Neumann Laplacian is defined through its associated quadratic form. As such, the self-adjoint property and the domain of its square root come out directly from the definition.

Let Q be a non-negative, densely defined, and closed sesquilinear form on a complex Hilbert space H with domain $\mathcal{D}(Q)$. Then Q uniquely determines a non-negative and self-adjoint operator S such that $\mathcal{D}(S^{1/2}) = \mathcal{D}(Q)$ and

$$Q(u,v) = \langle S^{1/2}u,\ S^{1/2}v\rangle$$

for all $u, v \in \mathcal{D}(Q)$. (See Theorem 4.4.2 in [D95], to which we refer the reader for the necessary spectral theoretic background used in this paper.) For any subspace $L \subset \mathcal{D}(Q)$, let $\lambda(L)=\sup\{Q(u,u) \mid u\in L, \|u\|=1\}$. For any positive integer j, let

$$\lambda_j(Q) = \inf\{\lambda(L) \mid L \subset \mathcal{D}(Q), \dim(L) = j\}. \tag{2.1}$$

The resolvent set $\rho(S)$ of S consists of all $\lambda \in \mathbb{C}$ such that the operator $S - \lambda I \colon \mathcal{D}(S) \to H$ is both one-to-one and onto (and hence has a bounded inverse by the closed graph theorem). The spectrum $\sigma(S)$, the complement of $\rho(S)$ in $\mathbb{C}$, is a non-empty closed subset of $[0,\ \infty)$. Its bottom $\inf \sigma(S)$ is given by $\lambda_1(Q)$. The essential spectrum $\sigma_e(S)$ is a closed subset of $\sigma(S)$ that consists of isolated eigenvalues of infinite multiplicity and accumulation points of the spectrum. It is empty if and only if $\lambda_j(Q) \to \infty$ as $j \to \infty$. In this case, $\lambda_j(Q)$ is the j^{th} eigenvalue of S, arranged in increasing order and repeated according to multiplicity. The bottom of the essential spectrum $\inf \sigma_e(T)$ is the limit of $\lambda_j(Q)$ as $j \to \infty$. (When $\sigma_e(S) = \emptyset$, we set $\inf \sigma_e(S) = \infty$.)

Let $T_k \colon H_k \to H_{k+1}$, $k = 1, 2$, be densely defined and closed operators on Hilbert spaces. Assume that $\mathcal{R}(T_1) \subset \mathcal{N}(T_2)$, where $\mathcal{R}$ and $\mathcal{N}$ denote the range and kernel of the operators. Let T_k^* be the Hilbert space adjoint of T_k, defined in the sense of Von Neumann by

$$\mathcal{D}(T_k^*) = \{u \in H_{k+1} \mid \exists C > 0, |\langle u, T_k v\rangle| \le C\|v\|, \forall v \in \mathcal{D}(T_k)\}$$

and

$$\langle T_k^* u, v\rangle = \langle u, T_k v\rangle, \quad \text{for all } u \in \mathcal{D}(T_k^*) \text{ and } v \in \mathcal{D}(T_k).$$

Then T_k^* is also densely defined and closed. Let

$$Q(u,v) = \langle T_1^* u, T_1^* v\rangle + \langle T_2 u, T_2 v\rangle$$

with its domain given by $\mathcal{D}(Q) = \mathcal{D}(T_1^*) \cap \mathcal{D}(T_2)$. The following proposition elucidates the above approach to the $\overline{\partial}$-Neumann Laplacian.

Proposition 2.1. *$Q(u,v)$ is a densely defined, closed, non-negative sesquilinear form. The associated self-adjoint operator $\Box$ is given by*

$$\mathcal{D}(\Box) = \{f \in H_2 \mid f \in \mathcal{D}(Q), T_2 f \in \mathcal{D}(T_2^*), T_1^* f \in \mathcal{D}(T_1)\}, \quad \Box = T_1 T_1^* + T_2^* T_2. \tag{2.2}$$

Proof. The closedness of Q follows easily from that of T_1 and T_2. The non-negativity is evident. We now prove that $\mathcal{D}(Q)$ is dense in H_2. Since $\mathcal{N}(T_2)^\perp = \overline{\mathcal{R}(T_2^*)} \subset \mathcal{N}(T_1^*)$ and

$$\mathcal{D}(T_2) = \mathcal{N}(T_2) \oplus \big(\mathcal{D}(T_2) \cap \mathcal{N}(T_2)^\perp\big),$$

we have

$$\mathcal{D}(Q) = \mathcal{D}(T_1^*) \cap \mathcal{D}(T_2) = \big(\mathcal{N}(T_2) \cap \mathcal{D}(T_1^*)\big) \oplus \big(\mathcal{D}(T_2) \cap \mathcal{N}(T_2)^\perp\big).$$

Since $\mathcal{D}(T_1^*)$ and $\mathcal{D}(T_2)$ are dense in H_2, $\mathcal{D}(Q)$ is dense in $\mathcal{N}(T_2) \oplus \mathcal{N}(T_2)^\perp = H_2$.

It follows from the above definition of $\Box$ that $f \in \mathcal{D}(\Box)$ if and only if $f \in \mathcal{D}(Q)$ and there exists a $g \in H_2$ such that

$$Q(u,f) = \langle u, g\rangle, \quad \text{for all } u \in \mathcal{D}(Q) \tag{2.3}$$

(cf. Lemma 4.4.1 in [D95]). Thus

$$\mathcal{D}(\Box) \supset \{f \in H_2 \mid f \in \mathcal{D}(Q), T_2 f \in \mathcal{D}(T_2^*), T_1^* f \in \mathcal{D}(T_1)\}.$$

We now prove the opposite containment. Suppose $f \in \mathcal{D}(\Box)$. For any $u \in \mathcal{D}(T_2)$, we write $u = u_1 + u_2 \in (\mathcal{N}(T_1^*) \cap \mathcal{D}(T_2)) \oplus \mathcal{N}(T_1^*)^\perp$. Note that $\mathcal{N}(T_1^*)^\perp \subset \mathcal{R}(T_2^*)^\perp = \mathcal{N}(T_2)$. It follows from (2.3) that

$$|\langle T_2 u, T_2 f\rangle| = |\langle T_2 u_1, T_2 f\rangle = |Q(u_1, f)| = |\langle u_1, g\rangle| \le \|u\| \cdot \|g\|.$$

Hence $T_2 f \in \mathcal{D}(T_2^*)$. The proof of $T_1^* f \in \mathcal{D}(T_1)$ is similar. For any $w \in \mathcal{D}(T_1^*)$, we write $w = w_1 + w_2 \in (\mathcal{N}(T_2) \cap \mathcal{D}(T_1^*)) \oplus \mathcal{N}(T_2)^\perp$. Note that $\mathcal{N}(T_2)^\perp = \overline{\mathcal{R}(T_2^*)} \subset \mathcal{N}(T_1^*)$. Therefore, by (2.3),

$$|\langle T_1^* w, T_1^* f\rangle| = |\langle T_1^* w_1, T_1^* f\rangle = |Q(w_1, f)| = |\langle w_1, g\rangle| \le \|w\| \cdot \|g\|.$$

Hence $T_1^* f \in \mathcal{D}(T_1^{**}) = \mathcal{D}(T_1)$. It follows from the definition of $\Box$ that for any $f \in \mathcal{D}(\Box)$ and $u \in \mathcal{D}(Q)$,

$$\begin{aligned}\langle \Box f, u\rangle &= \langle \Box^{1/2} f, \Box^{1/2} u\rangle = Q(f, u)\\ &= \langle T_1^* f, T_1^* u\rangle + \langle T_2 f, T_2 u\rangle = \langle (T_1 T_1^* + T_2^* T_2) f, u\rangle.\end{aligned}$$

Hence $\Box = T_1 T_1^* + T_2^* T_2$. □

The following proposition is well known (compare [H65], Theorem 1.1.2 and Theorem 1.1.4; [C83], Proposition 3; and [Sh92], Proposition 2.3). We provide a proof here for completeness.

Proposition 2.2. $\inf \sigma(\Box) > 0$ *if and only if* $\mathcal{R}(T_1) = \mathcal{N}(T_2)$ *and* $\mathcal{R}(T_2)$ *is closed.*

Proof. Assume $\inf \sigma(\Box) > 0$. Then 0 is in the resolvent set of $\Box$ and hence $\Box$ has a bounded inverse $G\colon H_2 \to \mathcal{D}(\Box)$. For any $u \in H_2$, write $u = T_1 T_1^* G u + T_2^* T_2 G u$. If $u \in \mathcal{N}(T_2)$, then $0 = (T_2 u,\ T_2 G u) = (T_2 T_2^* T_2 G u,\ T_2 G u) = (T_2^* T_2 G u,\ T_2^* T_2 G u)$. Hence $T_2^* T_2 G u = 0$ and $u = T_1 T_1^* G u$. Therefore, $\mathcal{R}(T_1) = \mathcal{N}(T_2)$. Similarly, $\mathcal{R}(T_2^*) = \mathcal{N}(T_1^*)$. Therefore T_2^* and hence T_2 have closed range. To prove the opposite implication, we write $u = u_1 + u_2 \in \mathcal{N}(T_2) \oplus \mathcal{N}(T_2)^\perp$, for any $u \in \mathcal{D}(Q)$. Note that $u_1, u_2 \in \mathcal{D}(Q)$. It follows from $\mathcal{N}(T_2) = \mathcal{R}(T_1)$ and the closed range property of T_2 that there exists a positive constant c such that $c\|u_1\|^2 \le \|T_1^* u_1\|^2$ and $c\|u_2\|^2 \le \|T_2 u_2\|^2$. Thus

$$c\|u\|^2 = c(\|u_1\|^2 + \|u_2\|^2) \le \|T_1^* u_1\|^2 + \|T_2 u_2\|^2 = Q(u, u).$$

Hence $\inf \sigma(\Box) \ge c > 0$ (cf. Theorem 4.3.1 in [D95]). □

Let $\mathcal{N}(Q) = \mathcal{N}(T_1^*) \cap \mathcal{N}(T_2)$. Note that when it is non-trivial, $\mathcal{N}(Q)$ is the eigenspace of the zero eigenvalue of $\Box$. When $\mathcal{R}(T_1)$ is closed, $\mathcal{N}(T_2) = \mathcal{R}(T_1) \oplus \mathcal{N}(Q)$. For a subspace $L \subseteq H_2$, denote by $P_{L^\perp}$ the orthogonal projection onto $L^\perp$ and $T_2|_{L^\perp}$ the restriction of T_2 to $L^\perp$. The next proposition clarifies and strengthens the second part of Lemma 2.1 in [Fu05].

Proposition 2.3. *The following statements are equivalent:*

1. $\inf \sigma_e(\Box) > 0$.
2. $\mathcal{R}(T_1)$ *and* $\mathcal{R}(T_2)$ *are closed and* $\mathcal{N}(Q)$ *is finite dimensional.*
3. *There exists a finite-dimensional subspace* $L \subset \mathcal{D}(T_1^*) \cap \mathcal{N}(T_2)$ *such that* $\mathcal{N}(T_2) \cap L^\perp = P_{L^\perp}(\mathcal{R}(T_1))$ *and* $\mathcal{R}(T_2|_{L^\perp})$ *is closed.*

Proof. We first prove (1) implies (2). Suppose $a = \inf \sigma_e(\Box) > 0$. If $\inf \sigma(\Box) > 0$, then $\mathcal{N}(Q)$ is trivial and (2) follows from Proposition 2.2. Suppose $\inf \sigma(\Box) = 0$. Then $\sigma(\Box) \cap [0, a)$ consists only of isolated points, all of which are eigenvalues of finite multiplicity of $\Box$ (cf. Theorem 4.5.2 in [D95]). Hence $\mathcal{N}(Q)$, the eigenspace of the eigenvalue 0, is finite dimensional. Choose a sufficiently small $c > 0$ so that $\sigma(\Box) \cap [0, c) = \{0\}$. By the spectral theorem for self-adjoint operators (cf. Theorem 2.5.1 in [D95]), there exists a finite regular Borel measure μ on $\sigma(\Box) \times \mathbb{N}$ and a unitary transformation $U\colon H_2 \to L^2(\sigma(\Box) \times \mathbb{N}, d\mu)$ such that $U \Box U^{-1} = M_x$, where

$M_x\varphi(x,n) = x\varphi(x,n)$ is the multiplication operator by x on $L^2(\sigma(\Box)\times\mathbb{N}, d\mu)$. Let $P_{\mathcal{N}(Q)}$ be the orthogonal projection onto $\mathcal{N}(Q)$. For any $f \in \mathcal{D}(Q)\cap\mathcal{N}(Q)^{\perp}$,

$$UP_{\mathcal{N}(Q)}f = \chi_{[0,c)}Uf = 0,$$

where $\chi_{[0,c)}$ is the characteristic function of $[0,c)$. Hence Uf is supported on $[c,\infty)$. Therefore,

$$Q(f,f) = \int_{\sigma(\Box)\times\mathbb{N}} x|Uf|^2\,d\mu \geq c\|Uf\|^2 = c\|f\|^2.$$

It then follows from Theorem 1.1.2 in [H65] that both T_1 and T_2 have closed range.

To prove (2) implies (1), we use Theorem 1.1.2 in [H65] in the opposite direction: There exists a positive constant c such that

$$c\|f\|^2 \leq Q(f,f), \quad \text{for all } f \in \mathcal{D}(Q)\cap\mathcal{N}(Q)^{\perp}. \tag{2.4}$$

Proving by contradiction, we assume $\inf\sigma_e(\Box) = 0$. Let ε be any positive number less than c. Since $L_{[0,\varepsilon)} = \mathcal{R}(\chi_{[0,\varepsilon)}(\Box))$ is infinite dimensional (cf. Lemma 4.1.4 in [D95]), there exists a non-zero $g \in L_{[0,\varepsilon)}$ such that $g \perp \mathcal{N}(Q)$. However,

$$Q(g,g) = \int_{\sigma(\Box)\times\mathbb{N}} x\chi_{[0,\varepsilon)}|Ug|^2\,d\mu \leq \varepsilon\|Ug\|^2 = \varepsilon\|g\|^2,$$

contradicting (2.4).

We do some preparations before proving the equivalence of (3) with (1) and (2). Let L be any finite-dimensional subspace of $\mathcal{D}(T_1^*)\cap\mathcal{N}(T_2)$. Let $H_2' = H_2\ominus L$. Let $T_2' = T_2\big|_{H_2'}$ and let $T_1^{*\prime} = T_1^*\big|_{H_2'}$. Then $T_2'\colon H_2' \to H_3$ and $T_1^{*\prime}\colon H_2' \to H_1$ are densely defined, closed operators. Let $T_1'\colon H_1 \to H_2'$ be the adjoint of $T_1^{*\prime}$. It follows from the definitions that $\mathcal{D}(T_1) \subset \mathcal{D}(T_1')$. The finite dimensionality of L implies the opposite containment. Thus, $\mathcal{D}(T_1) = \mathcal{D}(T_1')$. For any $f \in \mathcal{D}(T_1)$ and $g \in \mathcal{D}(T_1^{*\prime}) = \mathcal{D}(T_1^*)\cap L^{\perp}$,

$$\langle T_1'f, g\rangle = \langle f, T_1^{*\prime}g\rangle = \langle f, T_1^*g\rangle = \langle T_1f, g\rangle.$$

Hence $T_1' = P_{L^\perp}\circ T_1$ and $\mathcal{R}(T_1') = P_{L^\perp}(\mathcal{R}(T_1)) \subset \mathcal{N}(T_2')$. Let

$$Q'(f,g) = \langle T_1'^*f, T_1'^*g\rangle + \langle T_2'f, T_2'g\rangle$$

be the associated sesquilinear form on H_2' with $\mathcal{D}(Q') = \mathcal{D}(Q)\cap L^{\perp}$.

We are now in position to prove that (2) implies (3). In this case, we can take $L = \mathcal{N}(Q)$. By Theorem 1.1.2 in [H65], there exists a positive constant c such that

$$Q(f,f) = Q'(f,f) \geq c\|f\|^2, \quad \text{for all } f \in \mathcal{D}(Q').$$

We then obtain (3) by applying Proposition 2.2 to T_1', T_2', and $Q'(f,g)$.

Finally, we prove (3) implies (1). Applying Proposition 2.2 in the opposite direction, we know that there exists a positive constant c such that

$$Q(f,f) \geq c\|f\|^2, \quad \text{for all } f \in \mathcal{D}(Q)\cap L^{\perp}.$$

The rest of the proof follows the same lines of the above proof of the implication (2) $\Rightarrow$ (1), with $\mathcal{N}(Q)$ there replaced by L. $\Box$

We now recall the definition of the $\overline{\partial}$-Neumann Laplacian on a complex manifold. Let X be a complex hermitian manifold of dimension n. Let $C^\infty_{(0,q)}(X) = C^\infty(X, \Lambda^{0,q}T^*X)$ be the space of smooth $(0,q)$-forms on X. Let $\overline{\partial}_q \colon C^\infty_{(0,q)}(X) \to C^\infty_{(0,q+1)}(X)$ be the composition of the exterior differential operator and the projection onto $C^\infty_{(0,q+1)}(X)$.

Let Ω be a domain in X. For $u, v \in C^\infty_{(0,q)}(X)$, let $\langle u, v\rangle$ be the point-wise inner product of u and v, and let

$$\langle\langle u, v\rangle\rangle_\Omega = \int_\Omega \langle u, v\rangle dV$$

be the inner product of u and v over Ω. Let $L^2_{(0,q)}(\Omega)$ be the completion of the space of compactly supported forms in $C^\infty_{(0,q)}(\Omega)$ with respect to the above inner product. The operator $\overline{\partial}_q$ has a closed extension on $L^2_{(0,q)}(\Omega)$. We also denote the closure by $\overline{\partial}_q$. Thus $\overline{\partial}_q \colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ is densely defined and closed. Let $\overline{\partial}^*_q$ be its adjoint. For $1 \le q \le n-1$, let

$$Q_q(u, v) = \langle\langle \overline{\partial}_q u, \overline{\partial}_q v\rangle\rangle_\Omega + \langle\langle \overline{\partial}^*_{q-1} u, \overline{\partial}^*_{q-1} v\rangle\rangle_\Omega$$

be the sesquilinear form on $L^2_{(0,q)}(\Omega)$ with domain $\mathcal{D}(Q_q) = \mathcal{D}(\overline{\partial}_q) \cap \mathcal{D}(\overline{\partial}^*_{q-1})$. The self-adjoint operator $\Box_q$ associated with Q_q is called *the $\overline{\partial}$-Neumann Laplacian* on $L^2_{(0,q)}(\Omega)$. It is an elliptic operator with non-coercive boundary conditions [KN65].

The Dolbeault and L^2-cohomology groups on Ω are defined respectively by

$$H^{0,q}(\Omega) = \frac{\{f \in C^\infty_{(0,q)}(\Omega) \mid \overline{\partial}_q f = 0\}}{\{\overline{\partial}_{q-1} g \mid g \in C^\infty_{(0,q-1)}(\Omega)\}} \quad \text{and} \quad \widetilde{H}^{0,q}(\Omega) = \frac{\{f \in L^2_{(0,q)}(\Omega) \mid \overline{\partial}_q f = 0\}}{\{\overline{\partial}_{q-1} g \mid g \in L^2_{(0,q-1)}(\Omega)\}}.$$

These cohomology groups are in general not isomorphic. For example, when a complex variety is deleted from Ω, the L^2-cohomology group remains the same but the Dolbeault cohomology group could change from trivial to infinite-dimensional. As noted in the paragraph preceding Proposition 2.3, when $\mathcal{R}(\overline{\partial}_{q-1})$ is closed in $L^2_{(0,q)}(\Omega)$, $\widetilde{H}^{0,q}(\Omega) \cong \mathcal{N}(\Box_q)$. We refer the reader to [De] for an extensive treatise on the subject and to [H65] and [O82] for results relating these cohomology groups.

3. Positivity of the spectrum and essential spectrum

Laufer proved in [L75] that for any open subset of a Stein manifold, if a Dolbeault cohomology group is finite dimensional, then it is trivial. In this section, we establish the following L^2-analogue of this result on a bounded domain in a Stein manifold:

Theorem 3.1. *Let $\Omega \subset\subset X$ be a domain in a Stein manifold X with C^1 boundary. Let $\Box_q$, $1 \le q \le n-1$, be the $\overline{\partial}$-Neumann Laplacian on $L^2_{(0,q)}(\Omega)$. Assume that $\mathcal{N}(\Box_q) \subset W^1(\Omega)$. Then* $\inf \sigma(\Box_q) > 0$ *if and only if* $\inf \sigma_e(\Box_q) > 0$.

The proof of Theorem 3.1 follows the same line of arguments as Laufer's. We provide the details below.

Let $H^\infty(\Omega)$ be the space of bounded holomorphic functions on Ω. For any $f \in H^\infty(\Omega)$, let M_f be the multiplication operator by f:

$$M_f\colon L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega), \qquad M_f(u) = fu.$$

Then M_f induces an endomorphism on $\widetilde{H}^{0,q}(\Omega)$. Let $\mathcal{I}$ be set of all holomorphic functions $f \in H^\infty(\Omega)$ such that $M_f = 0$ on $\widetilde{H}^{0,q}(\Omega)$. Evidently, $\mathcal{I}$ is an ideal of $H^\infty(\Omega)$. Assume $\inf \sigma_e(\Box_q) > 0$. To show that $\widetilde{H}^{0,q}(\Omega)$ is trivial, it suffices to show that $1 \in \mathcal{I}$.

Lemma 3.2. *Let ξ be a holomorphic vector field on X and let $f \in \mathcal{I}$. Then $\xi(f) \in \mathcal{I}$.*

Proof. Let $D = \xi \lrcorner \partial\colon C^\infty_{(0,q)}(\Omega) \to C^\infty_{(0,q)}(\Omega)$, where $\lrcorner$ denotes the contraction operator. It is easy to check that D commute with the $\overline{\partial}$ operator. Therefore, D induces an endomorphism on $\widetilde{H}^{0,q}(\Omega)$. (Recall that under the assumption, $\widetilde{H}^{0,q}(\Omega) \cong \mathcal{N}(\Box_q) \subset W^1(\Omega)$.) For any $u \in \mathcal{N}(\Box_q)$,

$$D(fu) - fD(u) = \xi \lrcorner \partial(fu) - f\xi \lrcorner \partial u = \xi(f)u.$$

Notice that Ω is locally starlike near the boundary. Using partition of unity and the Friedrichs Lemma, we obtain $[D(fu)] = 0$. Therefore, $[\xi(f)u] = [D(fu)] - [fD(u)] = [0]$. $\Box$

We now return to the proof of the theorem. Let $F = (f_1, \ldots, f_{n+1})\colon X \to \mathbb{C}^{2n+1}$ be a proper embedding of X into $\mathbb{C}^{2n+1}$ (cf. Theorem 5.3.9 in [H91]). Since Ω is relatively compact in X, $f_j \in H^\infty(\Omega)$. For any f_j, let $P_j(\lambda)$ be the characteristic polynomial of $M_{f_j}\colon \widetilde{H}^{0,q}(\Omega) \to \widetilde{H}^{0,q}(\Omega)$. By the Cayley-Hamilton theorem, $P_j(M_{f_j}) = 0$ (cf. Theorem 2.4.2 in [HJ85]). Thus $P_j(f_j) \in \mathcal{I}$.

The number of points in the set $\{(\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}) \in \mathbb{C}^{2n+1} \mid P_j(\lambda_j) = 0, 1 \le j \le 2n+1\}$ is finite. Since $F\colon X \to \mathbb{C}^{2n+1}$ is one-to-one, the number of common zeroes of $P_j(f_j(z))$, $1 \le j \le 2n+1$, on X is also finite. Denote these zeroes by z^k, $1 \le k \le N$. For each z^k, let g_k be a function in $\mathcal{I}$ whose vanishing order at z^k is minimal. (Since $P_j(f_j) \in \mathcal{I}$, $g_k \not\equiv 0$.) We claim that $g_k(z^k) \neq 0$. Suppose otherwise $g_k(z^k) = 0$. Since there exists a holomorphic vector field ξ on X with any prescribed holomorphic tangent vector at any given point (cf. Corollary 5.6.3 in [H91]), one can find an appropriate choice of ξ so that $\xi(g_j)$ vanishes to lower order at z^k. According to Lemma 3.2, $\xi(g_j) \in \mathcal{I}$. We thus arrive at a contradiction.

Now we know that there are holomorphic functions, $P_j(f_j)$, $1 \le j \le 2n+1$, and g_k, $1 \le k \le N$, that have no common zeroes on X. It then follows that there exist holomorphic functions h_j on X such that

$$\sum P_j(f_j)h_j + \sum g_k h_k = 1.$$

(See, for example, Corollary 16 on p. 244 in [GR65], Theorem 7.2.9 in [H91], and Theorem 7.2.5 in [Kr01]. Compare also Theorem 2 in [Sk72].) Since $P_j(f_j) \in$

$\mathcal{I}, g_k \in \mathcal{I}$, and $h_j \in H^\infty(\Omega)$, we have $1 \in \mathcal{I}$. We thus conclude the proof of Theorem 3.1.

Remark. (1) Unlike the above-mentioned result of Laufer on the Dolbeault cohomology groups [L75], Theorem 3.1 is not expected to hold if the boundedness condition on Ω is removed (compare [W84]). It would be interesting to know whether Theorem 3.1 remains true if the assumption $\mathcal{N}(\Box_q) \subset W^1(\Omega)$ is dropped and whether it remains true for unbounded pseudoconvex domains.

(2) Notice that in the above proof, we use the fact that $\mathcal{R}(\overline{\partial}_{q-1})$ is closed, as a consequence of the assumption $\inf \sigma_e(\Box_q) > 0$ by Proposition 2.3. It is well known that for any infinite-dimensional Hilbert space H, there exists a subspace R of H such that H/R is finite dimensional but R is not closed. However, the construction of such a subspace usually involves Zorn's lemma (equivalently, the axiom of choice). It would be of interest to know whether there exists a domain Ω in a Stein manifold such that $\widetilde{H}^{0,q}(\Omega)$ is finite dimensional but $\mathcal{R}(\overline{\partial}_{q-1})$ is not closed.

(3) We refer the reader to [Sh09] for related results on the relationship between triviality and finite dimensionality of the L^2-cohomology groups using the $\overline{\partial}$-Cauchy problem. We also refer the reader to [B02] for a related result on embedded CR manifolds.

4. Hearing pseudoconvexity

The following theorem illustrates that one can easily determine pseudoconvexity from the spectrum of the $\overline{\partial}$-Neumann Laplacian.

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ such that* $\text{int}(\text{cl}(\Omega)) = \Omega$. *Then the following statements are equivalent:*

1. *Ω is pseudoconvex.*
2. $\inf \sigma(\Box_q) > 0$, *for all* $1 \le q \le n-1$.
3. $\inf \sigma_e(\Box_q) > 0$, *for all* $1 \le q \le n-1$.

The implication (1) $\Rightarrow$ (2) is a consequence of Hörmander's fundamental L^2-estimates of the $\overline{\partial}$-operator [H65], in light of Proposition 2.2, and it holds without the assumption $\text{int}(\text{cl}(\Omega)) = \Omega$. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are consequences of the sheaf cohomology theory dated back to Oka and Cartan (cf. [Se53, L66, Siu67, Br83, O88]). A elementary proof of (2) implying (1), as explained in [Fu05], is given below. The proof uses sheaf cohomology arguments in [L66]. When adapting Laufer's method to study the L^2-cohomology groups, one encounters a difficulty: While the restriction to the complex hyperplane of the smooth function resulting from the sheaf cohomology arguments for the Dolbeault cohomology groups is well defined, the restriction of the corresponding L^2 function is not. This difficulty was overcome in [Fu05] by appropriately modifying the construction of auxiliary $(0,q)$-forms (see the remark after the proof for more elaborations on this point).

We now show that (2) implies (1). Proving by contradiction, we assume that Ω is not pseudoconvex. Then there exists a domain $\widetilde{\Omega} \supsetneqq \Omega$ such that every holomorphic function on Ω extends to $\widetilde{\Omega}$. Since $\operatorname{int}(\operatorname{cl}(\Omega)) = \Omega$, $\widetilde{\Omega} \setminus \operatorname{cl}(\Omega)$ is non-empty. After a translation and a unitary transformation, we may assume that the origin is in $\widetilde{\Omega} \setminus \operatorname{cl}(\Omega)$ and there is a point z^0 in the intersection of z_n-plane with Ω that is in the same connected component of the intersection of the z_n-plane with $\widetilde{\Omega}$.

Let m be a positive integer (to be specified later). Let $k_q = n$. For any $\{k_1, \ldots, k_{q-1}\} \subset \{1, 2, \ldots, n-1\}$, we define

$$u(k_1, \ldots, k_q) = \frac{(q-1)!(\bar{z}_{k_1} \cdots \bar{z}_{k_q})^{m-1}}{r_m^q} \sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} d\bar{z}_{k_1} \wedge \cdots \wedge \widehat{d\bar{z}_{k_j}} \wedge \cdots \wedge d\bar{z}_{k_q}, \tag{4.1}$$

where $r_m = |z_1|^{2m} + \cdots + |z_n|^{2m}$. As usual, $\widehat{d\bar{z}_{k_j}}$ indicates the deletion of $d\bar{z}_{k_j}$ from the wedge product. Evidently, $u(k_1, \ldots, k_q) \in L^2_{(0,q-1)}(\Omega)$ is a smooth form on $\mathbb{C}^n \setminus \{0\}$. Moreover, $u(k_1, \ldots, k_q)$ is skew-symmetric with respect to the indices $(k_1, \ldots, k_{q-1})$. In particular, $u(k_1, \ldots, k_q) = 0$ when two k_j's are identical.

We now fix some notional conventions. Let $K = (k_1, \ldots, k_q)$ and J a collection of indices from $\{k_1, \ldots, k_q\}$. Write $d\bar{z}_K = d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}$, $\bar{z}_K^{m-1} = (\bar{z}_{k_1} \cdots \bar{z}_{k_q})^{m-1}$, and $\widetilde{d\bar{z}_{k_j}} = d\bar{z}_{k_1} \wedge \cdots \wedge \widehat{d\bar{z}_{k_j}} \wedge \ldots \wedge d\bar{z}_{k_q}$. Denote by $(k_1, \ldots, k_q \mid J)$ the tuple of remaining indices after deleting those in J from $(k_1, \ldots, k_q)$. For example, $(2, 5, 3, 1 \mid (4, 1, 6 \mid 4, 6)) = (2, 5, 3)$.

It follows from a straightforward calculation that

$$\begin{aligned}
\overline{\partial} u(k_1, \ldots, k_q) &= -\frac{q! m \bar{z}_K^{m-1}}{r_m^{q+1}} \Big(r_m d\bar{z}_K + \Big(\sum_{\ell=1}^{n} \bar{z}_\ell^{m-1} z_\ell^m d\bar{z}_\ell \Big) \wedge \Big(\sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} \widetilde{d\bar{z}_{k_j}} \Big) \Big) \\
&= -\frac{q! m \bar{z}_K^{m-1}}{r_m^{q+1}} \sum_{\substack{\ell=1 \\ \ell \neq k_1, \ldots, k_q}}^{n} z_\ell^m \bar{z}_\ell^{m-1} \Big(\bar{z}_\ell d\bar{z}_K + d\bar{z}_\ell \wedge \sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} \widetilde{d\bar{z}_{k_j}} \Big) \\
&= m \sum_{\ell=1}^{n-1} z_\ell^m u(\ell, k_1, \ldots, k_q).
\end{aligned} \tag{4.2}$$

It follows that $u(1, \ldots, n)$ is a $\overline{\partial}$-closed $(0, n-1)$-form.

By Proposition 2.2, we have $\mathcal{R}(\overline{\partial}_{q-1}) = \mathcal{N}(\overline{\partial}_q)$ for all $1 \leq q \leq n-1$. We now solve the $\overline{\partial}$-equations inductively, using $u(1, \ldots, n)$ as initial data. Let $v \in L^2_{(0,n-2)}(\Omega)$ be a solution to $\overline{\partial} v = u(1, \ldots, n)$. For any $k_1 \in \{1, \ldots, n-1\}$, define

$$w(k_1) = -m z_{k_1}^m v + (-1)^{1+k_1} u(1, \ldots, n \mid k_1).$$

Then it follows from (4.2) that $\overline{\partial} w(k_1) = 0$. Let $v(k_1) \in L^2_{(0,n-3)}(\Omega)$ be a solution of $\overline{\partial} v(k_1) = w(k_1)$.

Suppose for any $(q-1)$-tuple $K' = (k_1, \dots, k_{q-1})$ of integers from $\{1, \dots, n-1\}$, $q \geq 2$, we have constructed $v(K') \in L^2_{(0,n-q-1)}(\Omega)$ such that it is skew-symmetric with respect to the indices and satisfies

$$\overline{\partial} v(K') = m \sum_{j=1}^{q-1} (-1)^j z_{k_j}^m v(K' \mid k_j) + (-1)^{q+|K'|} u(1, \dots, n \mid K') \tag{4.3}$$

where $|K'| = k_1 + \cdots + k_{q-1}$ as usual. We now construct a $(0, n-q-2)$-forms $v(K)$ satisfying (4.3) for any q-tuple $K = (k_1, \dots, k_q)$ of integers from $\{1, \dots, n-1\}$ (with K' replaced by K). Let

$$w(K) = m \sum_{j=1}^{q} (-1)^j z_{k_j}^m v(K \mid k_j) + (-1)^{q+|K|} u(1, \dots, n \mid K).$$

Then it follows from (4.2) that

$$\begin{aligned}
&\overline{\partial} w(K) \\
&= m \sum_{j=1}^{q} (-1)^j z_{k_j}^m \overline{\partial} v(K \mid k_j) + (-1)^{q+|K|} \overline{\partial} u(1, \dots, n \mid K) \\
&= m \sum_{j=1}^{q} (-1)^j z_{k_j}^m \Big(m \sum_{1 \leq i < j} (-1)^i z_{k_i}^m v(K \mid k_j, k_i) + m \sum_{j < i \leq q} (-1)^{i-1} z_{k_i}^m v(K \mid k_j, k_i) \\
&\qquad - (-1)^{q+|K|-k_j} u(1, \dots, n \mid (K \mid k_j)) \Big) + (-1)^{q+|K|} \overline{\partial} u(1, \dots, n \mid K) \\
&= (-1)^{q+|K|} \big(-m \sum_{j=1}^{q} (-1)^{j-k_j} z_{k_j}^m u(1, \dots, n \mid (K \mid k_j)) + \overline{\partial} u(1, \dots, n \mid K) \big) \\
&= (-1)^{q+|K|} \big(-m \sum_{j=1}^{q} z_{k_j}^m u(k_j, (1, \dots, n \mid K)) + \overline{\partial} u(1, \dots, n \mid K) \big) = 0.
\end{aligned}$$

Therefore, by the hypothesis, there exists a $v(K) \in L^2_{(0,n-q-2)}(\Omega)$ such that $\overline{\partial} v(K) = w(K)$. Since $w(K)$ is skew-symmetric with respect to indices K, we may also choose a likewise $v(K)$. This then concludes the inductive step.

Now let

$$F = w(1, \dots, n-1) = m \sum_{j=1}^{n-1} z_j^m v(1, \dots, j, \dots, n-1) - (-1)^{n + \frac{n(n-1)}{2}} u(n),$$

where $u(n) = -\bar{z}_n^m / r_m$, as given by (4.1). Then $F(z) \in L^2(\Omega)$ and $\overline{\partial} F(z) = 0$. By the hypothesis, $F(z)$ has a holomorphic extension to $\widetilde{\Omega}$. We now restrict $F(z)$ to the coordinate hyperplane $z' = (z_1, \dots, z_{n-1}) = 0$. Notice that so far we only choose the $v(K)$'s and $w(K)$'s from L^2-spaces. The restriction to the coordinate hyperplane $z' = 0$ is not well defined.

To overcome this difficulty, we choose $m > 2(n-1)$. For sufficiently small $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ \int_{\{|z'|<\varepsilon\}\cap\Omega} \left|\left(F + (-1)^{n+\frac{n(n-1)}{2}} u(n)\right)(\delta z', z_n))\right|^2 dV(z) \right\}^{1/2}$$

$$\leq m\delta^m \varepsilon^m \sum_{j=1}^{n-1} \left\{ \int_{\{|z'|<\varepsilon\}\cap\Omega} |v(1,\ldots,\hat{j},\ldots,n-1)(\delta z', z_n)|^2 dV(z) \right\}^{1/2}$$

$$\leq m\delta^{m-2(n-1)} \varepsilon^m \sum_{j=1}^{n-1} \|v(1,\ldots,\hat{j},\ldots,n-1)\|_{L^2(\Omega)}.$$

Letting $\delta \to 0$, we then obtain

$$F(0, z_n) = -(-1)^{n+\frac{n(n-1)}{2}} u(n)(0, z_n) = (-1)^{n+\frac{n(n-1)}{2}} z_n^{-m}.$$

for z_n near z_n^0. (Recall that $z^0 \in \Omega$ is in the same connected component of $\{z' = 0\} \cap \widetilde{\Omega}$ as the origin.) This contradicts the analyticity of F near the origin. We therefore conclude the proof of Theorem 4.1.

Remark. (1) The above proof of the implication (2) $\Rightarrow$ (1) uses only the fact that the L^2-cohomology groups $\widetilde{H}^{0,q}(\Omega)$ are trivial for all $1 \leq q \leq n-1$. Under the (possibly) stronger assumption $\inf \sigma(\Box_q) > 0$, $1 \leq q \leq n-1$, the difficulty regarding the restriction of the L^2 function to the complex hyperplane in the proof becomes superficial. In this case, the $\overline{\partial}$-Neumannn Laplacian $\Box_q$ has a bounded inverse. The interior ellipticity of the $\overline{\partial}$-complex implies that one can in fact choose the forms $v(K)$ and $w(K)$ to be smooth inside Ω, using the canonical solution operator to the $\overline{\partial}$-equation. Therefore, in this case, the restriction to $\{z' = 0\} \cap \Omega$ is well defined. Hence one can choose $m = 1$. This was indeed the choice in [L66], where the forms involved are smooth and the restriction posts no problem. It is interesting to note that by having the freedom to choose m sufficiently large, one can leave out the use of interior ellipticity. Also, the freedom to choose m becomes crucial when one proves an analogue of Theorem 4.1 for the Kohn Laplacian because the $\overline{\partial}_b$-complex is no longer elliptic. The construction of $u(k_1, \ldots, k_q)$ in (4.1) with the exponent m was introduced in [Fu05] to handle this difficulty.

(2) One can similarly give a proof of the implication (3) $\Rightarrow$ (1). Indeed, the above proof can be easily modified to show that the finite dimensionality of $\widetilde{H}^{0,q}(\Omega)$, $1 \leq q \leq n-1$, implies the pseudoconvexity of Ω. In this case, the $u(K)$'s are defined by

$$u(k_1, \ldots, k_q) = \frac{(\alpha + q - 1)! \bar{z}_n^{m\alpha} (\bar{z}_{k_1} \cdots \bar{z}_{k_q})^{m-1}}{r_m^{\alpha+q}} \times \sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} d\bar{z}_{k_1} \wedge \cdots \wedge \widehat{d\bar{z}_{k_j}} \wedge \cdots \wedge d\bar{z}_{k_q},$$

where α is any non-negative integers. One now fixes a choice of $m > 2(n-1)$ and let α runs from 0 to N for a sufficiently large N, depending on the dimensions of the L^2-cohomology groups. We refer the reader to [Fu05] for details.

(3) As noted in Sections 2 and 3, unlike the Dolbeault cohomology case, one cannot remove the assumption $\operatorname{int}(\operatorname{cl}(\Omega)) = \Omega$ or the boundedness condition on Ω from Theorem 4.1. For example, a bounded pseudoconvex domain in $\mathbb{C}^n$ with a complex analytic variety removed still satisfies condition (2) in Theorem 3.1.

(4) As in [L66], Theorem 4.1 remains true for a Stein manifold. More generally, as a consequence of Andreotti-Grauert's theory [AG62], the q-convexity of a bounded domain Ω in a Stein manifold such that $\operatorname{int}(\operatorname{cl}(\Omega)) = \Omega$ is characterized by $\inf \sigma(\Box_k) > 0$ or $\inf \sigma_e(\Box_k) > 0$ for all $q \le k \le n-1$.

(5) It follows from Theorem 3.1 in [H04] that for a domain Ω in a complex hermitian manifold of dimension n, if $\inf \sigma_e(\Box_q) > 0$ for some q between 1 and $n-1$, then wherever the boundary is C^3-smooth, its Levi-form cannot have exactly $n-q-1$ positive and q negative eigenvalues. A complete characterization of a domain in a complex hermitian manifold, in fact, even in $\mathbb{C}^n$, that has $\inf \sigma_e(\Box_q) > 0$ or $\inf \sigma(\Box_q) > 0$ is unknown.

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Compactness Estimates for the $\overline{\partial}$-Neumann Problem in Weighted L^2-spaces

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Dedicated to Linda Rothschild

Abstract. In this paper we discuss compactness estimates for the $\overline{\partial}$-Neumann problem in the setting of weighted L^2-spaces on $\mathbb{C}^n$. For this purpose we use a version of the Rellich-Lemma for weighted Sobolev spaces.

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Keywords. $\overline{\partial}$-Neumann problem, Sobolev spaces, compactness.

1. Introduction

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. We consider the $\overline{\partial}$-complex

$$L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\overline{\partial}} 0\,,$$

where $L^2_{(0,q)}(\Omega)$ denotes the space of $(0,q)$-forms on Ω with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial}\left(\sum_J{}' a_J\, d\overline{z}_J\right) = \sum_{j=1}^n \sum_J{}' \frac{\partial a_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J,$$

where $\sum'$ means that the sum is only taken over strictly increasing multi-indices J.

The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right-hand side belongs to $L^2_{(0,q+1)}(\Omega)$. So $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ denoted by $\overline{\partial}^*$.

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The complex Laplacian $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}$ acts as an unbounded selfadjoint operator on $L^2_{(0,q)}(\Omega),\ 1 \leq q \leq n$, it is surjective and therefore has a continuous inverse, the $\overline{\partial}$-Neumann operator N_q. If v is a $\overline{\partial}$-closed $(0,q+1)$-form, then $u = \overline{\partial}^*\, N_{q+1}v$ provides the canonical solution to $\overline{\partial}u = v$, namely the one orthogonal to the kernel of $\overline{\partial}$ and so the one with minimal norm (see for instance [ChSh]).

A survey of the L^2-Sobolev theory of the $\overline{\partial}$-Neumann problem is given in [BS].

The question of compactness of N_q is of interest for various reasons. For example, compactness of N_q implies global regularity in the sense of preservation of Sobolev spaces [KN]. Also, the Fredholm theory of Toeplitz operators is an immediate consequence of compactness in the $\overline{\partial}$-Neumann problem [HI], [CD]. There are additional ramifications for certain C^*-algebras naturally associated to a domain in $\mathbb{C}^n$ [SSU]. Finally, compactness is a more robust property than global regularity – for example, it localizes, whereas global regularity does not – and it is generally believed to be more tractable than global regularity.

A thorough discussion of compactness in the $\overline{\partial}$-Neumann problem can be found in [FS1] and [FS2].

The study of the $\overline{\partial}$-Neumann problem is essentially equivalent to the study of the canonical solution operator to $\overline{\partial}$:

The $\overline{\partial}$-Neumann operator N_q is compact from $L^2_{(0,q)}(\Omega)$ to itself if and only if the canonical solution operators

$$\overline{\partial}^* N_q : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q-1)}(\Omega) \quad \text{and} \quad \overline{\partial}^* N_{q+1} : L^2_{(0,q+1)}(\Omega) \longrightarrow L^2_{(0,q)}(\Omega)$$

are compact.

Not very much is known in the case of unbounded domains. In this paper we continue the investigations of [HaHe] concerning existence and compactness of the canonical solution operator to $\overline{\partial}$ on weighted L^2-spaces over $\mathbb{C}^n$, where we applied ideas which were used in the spectral analysis of the Witten Laplacian in the real case, see [HeNi].

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function and define the space

$$L^2(\mathbb{C}^n, \varphi) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \ : \ \int_{\mathbb{C}^n} |f|^2\, e^{-\varphi}\, d\lambda < \infty\},$$

where λ denotes the Lebesgue measure, the space $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$ and the space $L^2_{(0,2)}(\mathbb{C}^n, \varphi)$ of $(0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$. Let

$$\langle f, g\rangle_\varphi = \int_{\mathbb{C}^n} f\,\overline{g} e^{-\varphi}\, d\lambda$$

denote the inner product and

$$\|f\|^2_\varphi = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi}\, d\lambda$$

the norm in $L^2(\mathbb{C}^n, \varphi)$.

We consider the weighted $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n,\varphi) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,1)}(\mathbb{C}^n,\varphi) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,2)}(\mathbb{C}^n,\varphi),$$

where $\overline{\partial}^*_\varphi$ is the adjoint operator to $\overline{\partial}$ with respect to the weighted inner product. For $u = \sum_{j=1}^n u_j d\overline{z}_j \in \mathrm{dom}(\overline{\partial}^*_\varphi)$ one has

$$\overline{\partial}^*_\varphi u = -\sum_{j=1}^n \left(\frac{\partial}{\partial z_j} - \frac{\partial\varphi}{\partial z_j}\right) u_j.$$

The complex Laplacian on $(0,1)$-forms is defined as

$$\Box_\varphi := \overline{\partial}\,\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi\overline{\partial},$$

where the symbol $\Box_\varphi$ is to be understood as the maximal closure of the operator initially defined on forms with coefficients in $\mathcal{C}_0^\infty$, i.e., the space of smooth functions with compact support.

$\Box_\varphi$ is a selfadjoint and positive operator, which means that

$$\langle \Box_\varphi f, f\rangle_\varphi \geq 0 \ , \ \text{for } f \in \mathrm{dom}(\Box_\varphi).$$

The associated Dirichlet form is denoted by

$$Q_\varphi(f,g) = \langle \overline{\partial} f, \overline{\partial} g\rangle_\varphi + \langle \overline{\partial}^*_\varphi f, \overline{\partial}^*_\varphi g\rangle_\varphi,$$

for $f, g \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*_\varphi)$. The weighted $\overline{\partial}$-Neumann operator N_φ is – if it exists – the bounded inverse of $\Box_\varphi$.

There is an interesting connection between $\overline{\partial}$ and the theory of Schrödinger operators with magnetic fields, see for example [Ch], [B], [FS3] and [ChF] for recent contributions exploiting this point of view.

Here we use a Rellich-Lemma for weighted Sobolev spaces to establish compactness estimates for the $\overline{\partial}$-Neumann operator N_φ on $L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ and we use this to give a new proof of the main result of [HaHe] without spectral theory of Schrödinger operators.

2. Weighted basic estimates

In the weighted space $L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ we can give a simple characterization of $\mathrm{dom}(\overline{\partial}^*_\varphi)$:

Proposition 2.1. *Let $f = \sum f_j d\overline{z}_j \in L^2_{(0,1)}(\mathbb{C}^n,\varphi)$. Then $f \in \mathrm{dom}(\overline{\partial}^*_\varphi)$ if and only if*

$$\sum_{j=1}^n \left(\frac{\partial f_j}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} f_j\right) \in L^2(\mathbb{C}^n,\varphi).$$

Proof. Suppose first that $\sum_{j=1}^n \left(\frac{\partial f_j}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} f_j\right) \in L^2(\mathbb{C}^n, \varphi)$, which equivalently means that $e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}(f_j e^{-\varphi}) \in L^2(\mathbb{C}^n, \varphi)$. We have to show that there exists a constant C such that $|\langle \overline{\partial} g, f\rangle_\varphi| \le C\|g\|_\varphi$ for all $g \in \operatorname{dom}(\overline{\partial})$. To this end let $(\chi_R)_{R\in\mathbb{N}}$ be a family of radially symmetric smooth cutoff functions, which are identically one on $\mathbb{B}_R$, the ball with radius R, such that the support of χ_R is contained in $\mathbb{B}_{R+1}$, $\operatorname{supp}(\chi_R) \subset \mathbb{B}_{R+1}$, and such that furthermore all first-order derivatives of all functions in this family are uniformly bounded by a constant M. Then for all $g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$:

$$\langle \overline{\partial} g, \chi_R f\rangle_\varphi = \sum_{j=1}^n \left\langle \frac{\partial g}{\partial \overline{z}_j}, \chi_R f_j \right\rangle_\varphi = -\int \sum_{j=1}^n g \frac{\partial}{\partial \overline{z}_j}\left(\chi_R \overline{f}_j e^{-\varphi}\right) d\lambda,$$

by integration by parts, which in particular means

$$|\langle \overline{\partial} g, f\rangle_\varphi| = \lim_{R\to\infty} |\langle \overline{\partial} g, \chi_R f\rangle_\varphi| = \lim_{R\to\infty}\left|\int_{\mathbb{C}^n} \sum_{j=1}^n g \frac{\partial}{\partial \overline{z}_j}\left(\chi_R \overline{f}_j e^{-\varphi}\right) d\lambda\right|.$$

Now we use the triangle inequality, afterwards Cauchy–Schwarz, to get

$$\begin{aligned}
&\lim_{R\to\infty}\left|\int_{\mathbb{C}^n} \sum_{j=1}^n g \frac{\partial}{\partial \overline{z}_j}\left(\chi_R \overline{f}_j e^{-\varphi}\right) d\lambda\right| \\
&\le \lim_{R\to\infty}\left|\int_{\mathbb{C}^n} \chi_R\, g \sum_{j=1}^n \frac{\partial}{\partial \overline{z}_j}\left(\overline{f}_j e^{-\varphi}\right) d\lambda\right| + \lim_{R\to\infty}\left|\int_{\mathbb{C}^n} \sum_{j=1}^n \overline{f}_j g \frac{\partial \chi_R}{\partial \overline{z}_j} e^{-\varphi}\, d\lambda\right| \\
&\le \lim_{R\to\infty} \|\chi_R\, g\|_\varphi \left\| e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right)\right\|_\varphi + M\|g\|_\varphi \|f\|_\varphi \\
&= \|g\|_\varphi \left\| e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right)\right\|_\varphi + M\|g\|_\varphi \|f\|_\varphi.
\end{aligned}$$

Hence by assumption,

$$|\langle \overline{\partial} g, f\rangle_\varphi| \le \|g\|_\varphi \left\| e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right)\right\|_\varphi + M\|g\|_\varphi \|f\|_\varphi \le C\|g\|_\varphi$$

for all $g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, and by density of $\mathcal{C}_0^\infty(\mathbb{C}^n)$ this is true for all $g \in \operatorname{dom}(\overline{\partial})$. Conversely, let $f \in \operatorname{dom}(\overline{\partial}_\varphi^*)$ and take $g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$. Then $g \in \operatorname{dom}(\overline{\partial})$ and

$$\begin{aligned}
\langle g, \overline{\partial}_\varphi^* f\rangle_\varphi = \langle \overline{\partial} g, f\rangle_\varphi &= \sum_{j=1}^n \left\langle \frac{\partial g}{\partial \overline{z}_j}, f_j\right\rangle_\varphi \\
&= -\left\langle g, \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right)\right\rangle_{L^2} = -\left\langle g, e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right)\right\rangle_\varphi.
\end{aligned}$$

Since $\mathcal{C}_0^\infty(\mathbb{C}^n)$ is dense in $L^2(\mathbb{C}^n, \varphi)$, we conclude that

$$\overline{\partial}_\varphi^* f = -e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f e^{-\varphi} \right),$$

which in particular implies that $e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j} (f_j e^{-\varphi}) \in L^2(\mathbb{C}^n, \varphi)$. □

The following lemma will be important for our considerations.

Lemma 2.2. *Forms with coefficients in* $\mathcal{C}_0^\infty(\mathbb{C}^n)$ *are dense in* $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}_\varphi^*)$ *in the graph norm* $f \mapsto (\|f\|_\varphi^2 + \|\overline{\partial} f\|_\varphi^2 + \|\overline{\partial}_\varphi^* f\|_\varphi^2)^{\frac{1}{2}}$.

Proof. First we show that compactly supported L^2-forms are dense in the graph norm. So let $\{\chi_R\}_{R\in\mathbb{N}}$ be a family of smooth radially symmetric cutoffs identically one on $\mathbb{B}_R$ and supported in $\mathbb{B}_{R+1}$, such that all first-order derivatives of the functions in this family are uniformly bounded in R by a constant M.
Let $f \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}_\varphi^*)$. Then, clearly, $\chi_R f \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}_\varphi^*)$ and $\chi_R f \to f$ in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ as $R \to \infty$. As observed in Proposition 2.1, we have

$$\overline{\partial}_\varphi^* f = -e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f_j e^{-\varphi} \right),$$

hence

$$\overline{\partial}_\varphi^* (\chi_R f) = -e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(\chi_R f_j e^{-\varphi} \right).$$

We need to estimate the difference of these expressions

$$\overline{\partial}_\varphi^* f - \overline{\partial}_\varphi^* (\chi_R f) = \overline{\partial}_\varphi^* f - \chi_R \overline{\partial}_\varphi^* f + \sum_{j=1}^n \frac{\partial \chi_R}{\partial z_j} f_j,$$

which is by the triangle inequality

$$\|\overline{\partial}_\varphi^* f - \overline{\partial}_\varphi^* (\chi_R f)\|_\varphi \leq \|\overline{\partial}_\varphi^* f - \chi_R \overline{\partial}_\varphi^* f\|_\varphi + M \sum_{j=1}^n \int_{\mathbb{C}^n \setminus \mathbb{B}_R} |f_j|^2 e^{-\varphi} \, d\lambda.$$

Now both terms tend to 0 as $R \to \infty$, and one can see similarly that also $\overline{\partial}(\chi_R f) \to \overline{\partial} f$ as $R \to \infty$.
So we have density of compactly supported forms in the graph norm, and density of forms with coefficients in $\mathcal{C}_0^\infty(\mathbb{C}^n)$ will follow by applying Friedrich's Lemma, see Appendix D in [ChSh], see also [Jo]. □

As in the case of bounded domains, the canonical solution operator to $\overline{\partial}$, which we denote by $\mathcal{S}_\varphi$, is given by $\overline{\partial}_\varphi^* N_\varphi$. Existence and compactness of N_φ and $\mathcal{S}_\varphi$ are closely related. At first, we notice that equivalent weight functions have the same properties in this regard.

Lemma 2.3. *Let φ_1 and φ_2 be two equivalent weights, i.e., $C^{-1}\|.\|_{\varphi_1} \leq \|.\|_{\varphi_2} \leq C\|.\|_{\varphi_1}$ for some $C > 0$. Suppose that $\mathcal{S}_{\varphi_2}$ exists. Then $\mathcal{S}_{\varphi_1}$ also exists and $\mathcal{S}_{\varphi_1}$ is compact if and only if $\mathcal{S}_{\varphi_2}$ is compact.*
An analog statement is true for the weighted $\overline{\partial}$-Neumann operator.

Proof. Let ι be the identity $\iota : L^2_{(0,1)}(\mathbb{C}^n, \varphi_1) \to L^2_{(0,1)}(\mathbb{C}^n, \varphi_2)$, $\iota f = f$, let j be the identity $j : L^2_{\varphi_2} \to L^2_{\varphi_1}$ and let furthermore P be the orthogonal projection onto $\ker(\overline{\partial})$ in $L^2_{\varphi_1}$. Since the weights are equivalent, ι and j are continuous, so if $\mathcal{S}_{\varphi_2}$ is compact, $j \circ \mathcal{S}_{\varphi_2} \circ \iota$ gives a solution operator on $L^2_{(0,1)}(\mathbb{C}^n, \varphi_1)$ that is compact. Therefore the canonical solution operator $\mathcal{S}_{\varphi_1} = P \circ j^{-1} \circ \mathcal{S}_{\varphi_2} \circ \iota$ is also compact. Since the problem is symmetric in φ_1 and φ_2, we are done.
The assertion for the Neumann operator follows by the identity

$$N_\varphi = \mathcal{S}_\varphi \mathcal{S}_\varphi^* + \mathcal{S}_\varphi^* \mathcal{S}_\varphi. \qquad \square$$

Note that whereas existence and compactness of the weighted $\overline{\partial}$-Neumann operator is invariant under equivalent weights by Lemma 2.3, regularity is not. For examples on bounded pseudoconvex domains, see for instance [ChSh], Chapter 6.

Now we suppose that the lowest eigenvalue λ_φ of the Levi-matrix

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{jk}$$

of φ satisfies

$$\liminf_{|z|\to\infty} \lambda_\varphi(z) > 0. \quad (^*)$$

Then, by Lemma 2.3, we may assume without loss of generality that $\lambda_\varphi(z) > \epsilon$ for some $\epsilon > 0$ and all $z \in \mathbb{C}^n$, since changing the weight function on a compact set does not influence our considerations. So we have the following basic estimate

Proposition 2.4. *For a plurisubharmonic weight function φ satisfying $(*)$, there is a $C > 0$ such that*

$$\|u\|_\varphi^2 \leq C(\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2)$$

for each $(0,1)$-form $u \in \operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}_\varphi^*)$.

Proof. By Lemma 2.2 and the assumption on φ it suffices to show that

$$\int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k \, e^{-\varphi} \, d\lambda \leq \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2,$$

for each $(0,1)$-form $u = \sum_{k=1}^n u_k \, d\overline{z}_k$ with coefficients $u_k \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, for $k = 1, \dots, n$.

For this purpose we set $\delta_k = \frac{\partial}{\partial z_k} - \frac{\partial \varphi}{\partial z_k}$ and get since

$$\overline{\partial} u = \sum_{j<k} \left(\frac{\partial u_j}{\partial \overline{z}_k} - \frac{\partial u_k}{\partial \overline{z}_j} \right) d\overline{z}_j \wedge d\overline{z}_k$$

that

$$\|\overline{\partial}u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2 = \int_{\mathbb{C}^n} \sum_{j<k} \left| \frac{\partial u_j}{\partial \overline{z}_k} - \frac{\partial u_k}{\partial \overline{z}_j} \right|^2 e^{-\varphi}\, d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n \delta_j u_j\, \overline{\delta_k u_k}\, e^{-\varphi}\, d\lambda$$
$$= \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \overline{z}_k} \right|^2 e^{-\varphi}\, d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left(\delta_j u_j\, \overline{\delta_k u_k} - \frac{\partial u_j}{\partial \overline{z}_k} \overline{\frac{\partial u_k}{\partial \overline{z}_j}} \right) e^{-\varphi}\, d\lambda$$
$$= \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \overline{z}_k} \right|^2 e^{-\varphi}\, d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left[\delta_j, \frac{\partial}{\partial \overline{z}_k} \right] u_j\, \overline{u}_k\, e^{-\varphi}\, d\lambda,$$

where we used the fact that for $f, g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ we have

$$\left\langle \frac{\partial f}{\partial \overline{z}_k}, g \right\rangle_\varphi = -\langle f, \delta_k g \rangle_\varphi.$$

Since

$$\left[\delta_j, \frac{\partial}{\partial \overline{z}_k} \right] = \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k},$$

and φ satisfies (*) we are done (see also [H]). □

Now it follows by Proposition 2.4 that there exists a uniquely determined $(0,1)$-form $N_\varphi u \in \operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}^*_\varphi)$ such that

$$\langle u, v \rangle_\varphi = Q_\varphi(N_\varphi u, v) = \langle \overline{\partial} N_\varphi u, \overline{\partial} v \rangle_\varphi + \langle \overline{\partial}^*_\varphi N_\varphi u, \overline{\partial}^*_\varphi v \rangle_\varphi,$$

and again by 2.4 that

$$\|\overline{\partial} N_\varphi u\|_\varphi^2 + \|\overline{\partial}^*_\varphi N_\varphi u\|_\varphi^2 \le C_1 \|u\|_\varphi^2$$

as well as

$$\|N_\varphi u\|_\varphi^2 \le C_2 (\|\overline{\partial} N_\varphi u\|_\varphi^2 + \|\overline{\partial}^*_\varphi N_\varphi u\|_\varphi^2) \le C_3 \|u\|_\varphi^2,$$

where $C_1, C_2, C_3 > 0$ are constants. Hence we get that N_φ is a continuous linear operator from $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into itself (see also [H] or [ChSh]).

3. Weighted Sobolev spaces

We want to study compactness of the weighted $\overline{\partial}$-Neumann operator N_φ. For this purpose we define weighted Sobolev spaces and prove, under suitable conditions, a Rellich-Lemma for these weighted Sobolev spaces. We will also have to consider their dual spaces, which already appeared in [BDH] and [KM].

Definition 3.1. *For $k \in \mathbb{N}$ let*

$$W^k(\mathbb{C}^n, \varphi) := \{ f \in L^2(\mathbb{C}^n, \varphi) \ : \ D^\alpha f \in L^2(\mathbb{C}^n, \varphi) \text{ for } |\alpha| \le k \},$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_{2n}} y_n}$ for $(z_1, \dots, z_n) = (x_1, y_1, \dots, x_n, y_n)$ with norm

$$\|f\|_{k,\varphi}^2 = \sum_{|\alpha| \le k} \|D^\alpha f\|_\varphi^2.$$

We will also need weighted Sobolev spaces with negative exponent. But it turns out that for our purposes it is more reasonable to consider the dual spaces of the following spaces.

Definition 3.2. *Let*

$$X_j = \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} \text{ and } Y_j = \frac{\partial}{\partial y_j} - \frac{\partial \varphi}{\partial y_j},$$

for $j = 1, \dots, n$ *and define*

$$W^1(\mathbb{C}^n, \varphi, \nabla\varphi) = \{f \in L^2(\mathbb{C}^n, \varphi) \ : X_j f, \ Y_j f \in L^2(\mathbb{C}^n, \varphi), j = 1, \dots, n\},$$

with norm

$$\|f\|^2_{\varphi,\nabla\varphi} = \|f\|^2_\varphi + \sum_{j=1}^n (\|X_j f\|^2_\varphi + \|Y_j f\|^2_\varphi).$$

In the next step we will analyze the dual space of $W^1(\mathbb{C}^n, \varphi, \nabla\varphi)$.

By the mapping $f \mapsto (f, X_j f, Y_j f)$, the space $W^1(\mathbb{C}^n, \varphi, \nabla\varphi)$ can be identified with a closed product of $L^2(\mathbb{C}^n, \varphi)$, hence each continuous linear functional L on $W^1(\mathbb{C}^n, \varphi, \nabla\varphi)$ is represented (in a non-unique way) by

$$\begin{aligned} L(f) &= \int_{\mathbb{C}^n} f(z) g_0(z) e^{-\varphi(z)} \, d\lambda(z) \\ &+ \sum_{j=1}^n \int_{\mathbb{C}^n} (X_j f(z) g_j(z) + Y_j f(z) h_j(z)) e^{-\varphi(z)} \, d\lambda(z), \end{aligned}$$

for some $g_j, h_j \in L^2(\mathbb{C}^n, \varphi)$.

For $f \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ it follows that

$$L(f) = \int_{\mathbb{C}^n} f(z) g_0(z) e^{-\varphi(z)} \, d\lambda(z) - \sum_{j=1}^n \int_{\mathbb{C}^n} f(z) \left(\frac{\partial g_j(z)}{\partial x_j} + \frac{\partial h_j(z)}{\partial y_j} \right) e^{-\varphi(z)} \, d\lambda(z).$$

Since $\mathcal{C}_0^\infty(\mathbb{C}^n)$ is dense in $W^1(\mathbb{C}^n, \varphi, \nabla\varphi)$ we have shown

Lemma 3.3. *Each element* $u \in W^{-1}(\mathbb{C}^n, \varphi, \nabla\varphi) := (W^1(\mathbb{C}^n, \varphi, \nabla\varphi))'$ *can be represented in a non-unique way by*

$$u = g_0 + \sum_{j=1}^n \left(\frac{\partial g_j}{\partial x_j} + \frac{\partial h_j}{\partial y_j} \right),$$

where $g_j, h_j \in L^2(\mathbb{C}^n, \varphi)$.

The dual norm $\|u\|_{-1,\varphi,\nabla\varphi} := \sup\{|u(f)| \ : \ \|f\|_{\varphi,\nabla\varphi} \leq 1\}$ *can be expressed in the form*

$$\|u\|^2_{-1,\varphi,\nabla\varphi} = \inf\{\|g_0\|^2 + \sum_{j=1}^n (\|g_j\|^2 + \|h_j\|^2),$$

where the infimum is taken over all families (g_j, h_j) *in* $L^2(\mathbb{C}^n, \varphi)$ *representing the functional* u (see for instance [T]).

In particular each function in $L^2(\mathbb{C}^n, \varphi)$ can be identified with an element of $W^{-1}(\mathbb{C}^n, \varphi, \nabla\varphi)$.

Proposition 3.4. *Suppose that the weight function satisfies*

$$\lim_{|z|\to\infty} (\theta|\nabla\varphi(z)|^2 + \triangle\varphi(z)) = +\infty,$$

for some $\theta \in (0,1)$, *where*

$$|\nabla\varphi(z)|^2 = \sum_{k=1}^n \left(\left|\frac{\partial\varphi}{\partial x_k}\right|^2 + \left|\frac{\partial\varphi}{\partial y_k}\right|^2 \right).$$

Then the embedding of $W^1(\mathbb{C}^n, \varphi, \nabla\varphi)$ *into* $L^2(\mathbb{C}^n, \varphi)$ *is compact.*

Proof. We adapt methods from [BDH] or [Jo], Proposition 6.2., or [KM]. For the vector fields X_j from 3.2 and their formal adjoints $X_j^* = -\frac{\partial}{\partial x_j}$ we have

$$(X_j + X_j^*)f = -\frac{\partial\varphi}{\partial x_j}\, f \text{ and } [X_j, X_j^*]f = -\frac{\partial^2\varphi}{\partial x_j^2}\, f,$$

for $f \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, and

$$\langle [X_j, X_j^*]f, f\rangle_\varphi = \|X_j^* f\|_\varphi^2 - \|X_j f\|_\varphi^2,$$

$$\|(X_j + X_j^*)f\|_\varphi^2 \leq (1 + 1/\epsilon)\|X_j f\|_\varphi^2 + (1+\epsilon)\|X_j^* f\|_\varphi^2$$

for each $\epsilon > 0$. Similar relations hold for the vector fields Y_j. Now we set

$$\Psi(z) = |\nabla\varphi(z)|^2 + (1+\epsilon)\triangle\varphi(z).$$

It follows that

$$\langle \Psi f, f\rangle_\varphi \leq (2 + \epsilon + 1/\epsilon) \sum_{j=1}^n (\|X_j f\|_\varphi^2 + \|Y_j f\|_\varphi^2).$$

Since $\mathcal{C}_0^\infty(\mathbb{C}^n)$ is dense in $W^1(\mathbb{C}^n, \varphi, \nabla\varphi)$ by definition, this inequality holds for all $f \in W^1(\mathbb{C}^n, \varphi, \nabla\varphi)$.

If $(f_k)_k$ is a sequence in $W^1(\mathbb{C}^n, \varphi, \nabla\varphi)$ converging weakly to 0, then $(f_k)_k$ is a bounded sequence in $W^1(\mathbb{C}^n, \varphi, \nabla\varphi)$ and our the assumption implies that

$$\Psi(z) = |\nabla\varphi(z)|^2 + (1+\epsilon)\triangle\varphi(z)$$

is positive in a neighborhood of ∞. So we obtain

$$\begin{aligned}
&\int_{\mathbb{C}^n} |f_k(z)|^2 e^{-\varphi(z)}\, d\lambda(z) \\
&\leq \int_{|z|<R} |f_k(z)|^2 e^{-\varphi(z)}\, d\lambda(z) + \int_{|z|\geq R} \frac{\Psi(z)|f_k(z)|^2}{\inf\{\Psi(z) \,:\, |z| \geq R\}}\, e^{-\varphi(z)}\, d\lambda(z) \\
&\leq C_{\varphi,R}\, \|f_k\|_{L^2(\mathbb{B}_R)}^2 + \frac{C_\epsilon\, \|f_k\|_{\varphi,\nabla\varphi}^2}{\inf\{\Psi(z) \,:\, |z| \geq R\}}.
\end{aligned}$$

Hence the assumption and the fact that the injection $W^1(\mathbb{B}_R) \hookrightarrow L^2(\mathbb{B}_R)$ is compact (see for instance [T]) show that a subsequence of $(f_k)_k$ tends to 0 in $L^2(\mathbb{C}^n, \varphi)$. □

Remark 3.5. *It follows that the adjoint to the above embedding, the embedding of* $L^2(\mathbb{C}^n, \varphi)$ *into* $(W^1(\mathbb{C}^n, \varphi, \nabla\varphi))' = W^{-1}(\mathbb{C}^n, \varphi, \nabla\varphi)$ *(in the sense of* 3.3*) is also compact.*

Remark 3.6. *Note that one does not need plurisubharmonicity of the weight function in Proposition* 3.4*. If the weight is plurisubharmonic, one can of course drop* θ *in the formulation of the assumption.*

4. Compactness estimates

The following Proposition reformulates the compactness condition for the case of a bounded pseudoconvex domain in $\mathbb{C}^n$, see [BS], [Str]. The difference to the compactness estimates for bounded pseudoconvex domains is that here we have to assume an additional condition on the weight function implying a corresponding Rellich-Lemma.

Proposition 4.1. *Suppose that the weight function* φ *satisfies* $(*)$ *and*

$$\lim_{|z|\to\infty} (\theta|\nabla\varphi(z)|^2 + \triangle\varphi(z)) = +\infty,$$

for some $\theta \in (0,1)$*, then the following statements are equivalent.*

1. *The* $\overline{\partial}$*-Neumann operator* $N_{1,\varphi}$ *is a compact operator from* $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ *into itself.*
2. *The embedding of the space* $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*_\varphi)$*, provided with the graph norm* $u \mapsto (\|u\|^2_\varphi + \|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)^{1/2}$*, into* $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ *is compact.*
3. *For every positive* ϵ *there exists a constant* C_ϵ *such that*
$$\|u\|^2_\varphi \le \epsilon(\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi) + C_\epsilon \|u\|^2_{-1,\varphi,\nabla\varphi},$$
for all $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*_\varphi)$.
4. *The operators*
$$\overline{\partial}^*_\varphi N_{1,\varphi} : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \cap \ker(\overline{\partial}) \longrightarrow L^2(\mathbb{C}^n, \varphi) \quad \text{and}$$
$$\overline{\partial}^*_\varphi N_{2,\varphi} : L^2_{(0,2)}(\mathbb{C}^n, \varphi) \cap \ker(\overline{\partial}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$$
are both compact.

Proof. First we show that (1) and (4) are equivalent: suppose that $N_{1,\varphi}$ is compact. For $f \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ it follows that

$$\|\overline{\partial}^*_\varphi N_{1,\varphi} f\|^2_\varphi \le \langle f, N_{1,\varphi} f\rangle_\varphi \le \epsilon \|f\|^2_\varphi + C_\epsilon \|N_{1,\varphi} f\|^2_\varphi$$

by Lemma 2 of [CD]. Hence $\overline{\partial}^*_\varphi N_{1,\varphi}$ is compact. Applying the formula

$$N_{1,\varphi} - (\overline{\partial}^*_\varphi N_{1,\varphi})^*(\overline{\partial}^*_\varphi N_{1,\varphi}) = (\overline{\partial}^*_\varphi N_{2,\varphi})(\overline{\partial}^*_\varphi N_{2,\varphi})^*,$$

(see for instance [ChSh]), we get that also $\overline{\partial}^*_\varphi N_{2,\varphi}$ is compact. The converse follows easily from the same formula.

Now we show (4) $\implies$ (3) $\implies$ (2) $\implies$ (1). We follow the lines of [Str], where the case of a bounded pseudoconvex domain is handled.

Assume (4): if (3) does not hold, then there exists $\epsilon_0 > 0$ and a sequence $(u_n)_n$ in $\mathrm{dom}\,(\overline{\partial}) \cap \mathrm{dom}\,(\overline{\partial}^*_\varphi)$ with $\|u_n\|_\varphi = 1$ and

$$\|u_n\|^2_\varphi \geq \epsilon_0(\|\overline{\partial}u_n\|^2_\varphi + \|\overline{\partial}^*_\varphi u_n\|^2_\varphi) + n\|u_n\|^2_{-1,\varphi,\nabla\varphi}$$

for each $n \geq 1$, which implies that $u_n \to 0$ in $W^{-1}_{(0,1)}(\mathbb{C}^n, \varphi, \nabla\varphi)$. Since u_n can be written in the form

$$u_n = (\overline{\partial}^*_\varphi N_{1,\varphi})^*\,\overline{\partial}^*_\varphi u_n + (\overline{\partial}^*_\varphi N_{2,\varphi})\,\overline{\partial}u_n,$$

(4) implies there exists a subsequence of $(u_n)_n$ converging in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ and the limit must be 0, which contradicts $\|u_n\|_\varphi = 1$.

To show that (3) implies (2) we consider a bounded sequence in $\mathrm{dom}\,(\overline{\partial}) \cap \mathrm{dom}\,(\overline{\partial}^*_\varphi)$. By 2.4 this sequence is also bounded in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. Now 3.4 implies that it has a subsequence converging in $W^{-1}_{(0,1)}(\mathbb{C}^n, \varphi, \nabla\varphi)$. Finally use (3) to show that this subsequence is a Cauchy sequence in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$, therefore (2) holds.

Assume (2): by 2.4 and the basic facts about $N_{1,\varphi}$, it follows that

$$N_{1,\varphi} : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \longrightarrow \mathrm{dom}\,(\overline{\partial}) \cap \mathrm{dom}\,(\overline{\partial}^*_\varphi)$$

is continuous in the graph topology, hence

$$N_{1,\varphi} : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \longrightarrow \mathrm{dom}\,(\overline{\partial}) \cap \mathrm{dom}\,(\overline{\partial}^*_\varphi) \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$$

is compact. □

Remark 4.2. *Suppose that the weight function φ is plurisubharmonic and that the lowest eigenvalue λ_φ of the Levi-matrix M_φ satisfies*

$$\lim_{|z|\to\infty} \lambda_\varphi(z) = +\infty\,. \quad (^{**})$$

This condition implies that $N_{1,\varphi}$ is compact [HaHe].

It also implies that the condition of the Rellich-Lemma 3.4 *is satisfied.*

This follows from the fact that we have for the trace $\mathrm{tr}(M_\varphi)$ of the Levi-matrix

$$\mathrm{tr}(M_\varphi) = \frac{1}{4}\triangle\varphi,$$

and since for any invertible $(n \times n)$-matrix T

$$\mathrm{tr}(M_\varphi) = \mathrm{tr}(TM_\varphi T^{-1}),$$

it follows that $\mathrm{tr}(M_\varphi)$ equals the sum of all eigenvalues of M_φ. Hence our assumption on the lowest eigenvalue λ_φ of the Levi-matrix implies that the assumption of Proposition 3.4 is satisfied.

In order to use Proposition 4.1 to show compactness of N_φ we still need

Proposition 4.3 (Gårding's inequality). *Let Ω be a smooth bounded domain. Then for any $u \in W^1(\Omega, \varphi, \nabla\varphi)$ with compact support in Ω*

$$\|u\|^2_{1,\varphi,\nabla\varphi} \le C(\Omega, \varphi)\left(\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi + \|u\|^2_\varphi\right).$$

Proof. The operator $-\Box_\varphi$ is strictly elliptic since its principal part equals the Laplacian. Now $-\Box_\varphi = -(\overline{\partial} \oplus \overline{\partial}^*_\varphi)^* \circ (\overline{\partial} \oplus \overline{\partial}^*_\varphi)$, so from general PDE theory follows that the system $\overline{\partial} \oplus \overline{\partial}^*_\varphi$ is elliptic. This is, because a differential operator P of order s is elliptic if and only if $(-1)^s P^* \circ P$ is strictly elliptic. So because of ellipticity, one has on each smooth bounded domain Ω the classical Gårding inequality

$$\|u\|^2_1 \le C(\Omega)\left(\|\overline{\partial}u\|^2 + \|\overline{\partial}^*_\varphi u\|^2 + \|u\|^2\right)$$

for any (0,1)-form u with coefficients in $\mathcal{C}^\infty_0$. But our weight φ is smooth on $\overline{\Omega}$, hence the weighted and unweighted L^2-norms on Ω are equivalent, and therefore

$$\begin{aligned}\|u\|^2_{1,\varphi,\nabla\varphi} &\le C_1(\|u\|^2_{1,\varphi} + \|u\|^2_\varphi) \le C_2(\|u\|^2_1 + \|u\|^2)\\ &\le C_3(\|\overline{\partial}u\|^2 + \|\overline{\partial}^*_\varphi u\|^2 + \|u\|^2) \le C_4(\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi + \|u\|^2_\varphi).\end{aligned}$$

□

We are now able to give a different proof of the main result in [HaHe].

Theorem 4.4. *Let φ be plurisubharmonic. If the lowest eigenvalue $\lambda_\varphi(z)$ of the Levi-matrix M_φ satisfies (**), then N_φ is compact.*

Proof. By Proposition 3.4 and Remark 4.2, it suffices to show a compactness estimate and use Proposition 4.1. Given $\epsilon > 0$ we choose $M \in \mathbb{N}$ with $1/M \le \epsilon/2$ and R such that $\lambda(z) > M$ whenever $|z| > R$. Let χ be a smooth cutoff function identically one on $\mathbb{B}_R$. Hence we can estimate

$$\begin{aligned}M\|f\|^2_\varphi \le &\sum_{j,k} \int_{\mathbb{C}^n\setminus\mathbb{B}_R} \frac{\partial^2\varphi}{\partial z_j \partial\overline{z}_k} f_j \overline{f}_k e^{-\varphi}\, d\lambda + M\|\chi f\|^2_\varphi\\ \le &Q_\varphi(f,f) + M\langle \chi f, f\rangle_\varphi\\ \le &Q_\varphi(f,f) + M\|\chi f\|_{1,\varphi,\nabla\varphi}\|f\|_{-1,\varphi,\nabla\varphi}\\ \le &Q_\varphi(f,f) + Ma\|\chi f\|^2_{1,\varphi,\nabla\varphi} + a^{-1}M\|f\|^2_{-1,\varphi,\nabla\varphi},\end{aligned}$$

where a is to be chosen a bit later. Now we apply Gårding's inequality 4.3 to the second term, so there is a constant C_R depending on R such that

$$M\|f\|^2_\varphi \le Q_\varphi(f,f) + MaC_R\left(Q_\varphi(f,f) + \|f\|^2_\varphi\right) + a^{-1}M\|f\|^2_{-1,\varphi,\nabla\varphi}.$$

By Proposition 2.4 and after increasing C_R we have

$$M\|f\|^2_\varphi \le Q_\varphi(f,f) + MaC_R Q_\varphi(f,f) + a^{-1}M\|f\|^2_{-1,\varphi,\nabla\varphi}.$$

Now choose a such that $aC_R \le \epsilon/2$, then

$$\|f\|^2_\varphi \le \epsilon Q_\varphi(f,f) + a^{-1}\|f\|^2_{-1,\varphi,\nabla\varphi}$$

and this estimate is equivalent to compactness by 4.1. □

Remark 4.5. *Assumption* (**) *on the lowest eigenvalue of* M_φ *is the analog of property* (P) *introduced by Catlin in* [Ca] *in case of bounded pseudoconvex domains. Therefore the proof is similar.*

Remark 4.6. *We mention that for the weight* $\varphi(z) = |z|^2$ *the* $\overline{\partial}$*-Neumann operator fails to be compact* (*see* [HaHe]), *but the condition*

$$\lim_{|z|\to\infty} (\theta|\nabla\varphi(z)|^2 + \triangle\varphi(z)) = +\infty$$

of the Rellich-Lemma is satisfied.

Remark 4.7. *Denote by* $W^m_{\text{loc}}(\mathbb{C}^n)$ *the space of functions which locally belong to the classical unweighted Sobolev space* $W^m(\mathbb{C}^n)$. *Suppose that* $\Box_\varphi v = g$ *and* $g \in W^m_{\text{loc}\,(0,1)}(\mathbb{C}^n)$. *Then* $v \in W^{m+2}_{\text{loc}\,(0,1)}(\mathbb{C}^n)$. *In particular, if there exists a weighted* $\overline{\partial}$*-Neumann operator* N_φ, *it maps* $\mathcal{C}^\infty_{(0,1)}(\mathbb{C}^n) \cap L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ *into itself.*

$\Box_\varphi$ is strictly elliptic, and the statement in fact follows from interior regularity of a general second-order elliptic operator. The reader can find more on elliptic regularity for instance in [Ev], Chapter 6.3.
An analog statement is true for $\mathcal{S}_\varphi$. If there exists a continuous canonical solution operator $\mathcal{S}_\varphi$, it maps $\mathcal{C}^\infty_{(0,1)}(\mathbb{C}^n) \cap L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into itself. This follows from ellipticity of $\overline{\partial}$.

Although $\Box_\varphi$ is strictly elliptic, the question whether $\mathcal{S}_\varphi$ is globally or exactly regular is harder to answer. This is, because our domain is not bounded and neither are the coefficients of $\Box_\varphi$. Only in a very special case the question is easy – this is, when A^2_φ (the weighted space of entire functions) is zero. In this case, there is only one solution operator to $\overline{\partial}$, namely the canonical one, and if $f \in W^k_{\varphi\,(0,1)}$ and $u = \mathcal{S}_\varphi f$, it follows that $\overline{\partial} D^\alpha u = D^\alpha f$, since $\overline{\partial}$ commutes with $\frac{\partial}{\partial x_j}$. Now $\mathcal{S}_\varphi$ is continuous, so $\|D^\alpha u\|_\varphi \leq C\|D^\alpha f\|_\varphi$, meaning that $u \in W^k_\varphi$. So in this case $\mathcal{S}_\varphi$ is a bounded operator from $W^k_{\varphi\,(0,1)} \to W^k_\varphi$.

Remark 4.8. *Let* $A^2_{(0,1)}(\mathbb{C}^n, \varphi)$ *denote the space of* $(0,1)$*-forms with holomorphic coefficients belonging to* $L^2(\mathbb{C}^n.\varphi)$.

We point out that assuming (**) *implies directly – without use of Sobolev spaces – that the embedding of the space*

$$A^2_{(0,1)}(\mathbb{C}^n, \varphi) \cap dom\,(\overline{\partial}^*_\varphi)$$

provided with the graph norm $u \mapsto (\|u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)^{1/2}$ *into* $A^2_{(0,1)}(\mathbb{C}^n, \varphi)$ *is compact. Compare* 4.1 (2).

For this purpose let $u \in A^2_{(0,1)}(\mathbb{C}^n, \varphi) \cap dom\,(\overline{\partial}^*_\varphi)$. Then we obtain from the proof of 2.4 that

$$\|\overline{\partial}^*_\varphi u\|^2_\varphi = \int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2\varphi}{\partial z_j \partial\overline{z}_k}\, u_j \overline{u}_k\, e^{-\varphi}\, d\lambda.$$

Let us for $u = \sum_{j=1}^n u_j \, d\overline{z}_j$ identify $u(z)$ with the vector $(u_1(z), \dots, u_n(z)) \in \mathbb{C}^n$. Then, if we denote by $\langle ., . \rangle$ the standard inner product in $\mathbb{C}^n$, we have

$$\langle u(z), u(z) \rangle = \sum_{j=1}^n |u_j(z)|^2$$

and

$$\langle M_\varphi u(z), u(z) \rangle = \sum_{j,k=1}^n \frac{\partial^2 \varphi(z)}{\partial z_j \partial \overline{z}_k} u_j(z) \overline{u_k(z)}.$$

Note that the lowest eigenvalue λ_φ of the Levi-matrix M_φ can be expressed as

$$\lambda_\varphi(z) = \inf_{u(z) \neq 0} \frac{\langle M_\varphi u(z), u(z) \rangle}{\langle u(z), u(z) \rangle}.$$

So we get

$$\begin{aligned} \int_{\mathbb{C}^n} \langle u, u \rangle e^{-\varphi} \, d\lambda &\leq \int_{\mathbb{B}_R} \langle u, u \rangle e^{-\varphi} \, d\lambda + [\inf_{\mathbb{C}^n \setminus \mathbb{B}_R} \lambda_\varphi(z)]^{-1} \int_{\mathbb{C}^n \setminus \mathbb{B}_R} \lambda_\varphi(z) \, \langle u, u \rangle e^{-\varphi} \, d\lambda \\ &\leq \int_{\mathbb{B}_R} \langle u, u \rangle e^{-\varphi} \, d\lambda + [\inf_{\mathbb{C}^n \setminus \mathbb{B}_R} \lambda_\varphi(z)]^{-1} \int_{\mathbb{C}^n} \langle M_\varphi u, u \rangle e^{-\varphi} \, d\lambda. \end{aligned}$$

For a given $\epsilon > 0$ choose R so large that

$$[\inf_{\mathbb{C}^n \setminus \mathbb{B}_R} \lambda_\varphi(z)]^{-1} < \epsilon,$$

and use the fact that for Bergman spaces of holomorphic functions the embedding of $A^2(\mathbb{B}_{R_1})$ into $A^2(\mathbb{B}_{R_2})$ is compact for $R_2 < R_1$. So the desired conclusion follows.

Remark 4.9. *Part of the results, in particular Theorem* 4.4*, are taken from* [Ga]. *We finally mention that the methods used in this paper can also be applied to treat unbounded pseudoconvex domains with boundary, see* [Ga].

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Remarks on the Homogeneous Complex Monge-Ampère Equation

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Dedicated to Professor Linda Rothchild on the occasion of her 60th birthday

Abstract. We refine the arguments in [12] to show that the extended norm of Bedford-Taylor is in fact exact the same as the original Chern-Levine-Nirenberg intrinsic norm, thus provides a proof of the Chern-Levine-Nirenberg conjecture. The result can be generalized to deal with homogeneous Monge-Ampère equation on any complex manifold.

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This short note concerns the homogeneous complex Monge-Ampère equation arising from the Chern-Levine-Nirenberg holomorphic invariant norms in [9]. In [9], Chern-Levine-Nirenberg found close relationship of the intrinsic norms with the variational properties and regularity of the homogeneous complex Monge-Ampère equation. It is known that solutions to the homogeneous complex Monge-Ampère equation fail to be C^2 in general, since the equation is degenerate and the best regularity is $C^{1,1}$ by examples of Bedford-Forneass [2]. In a subsequential study undertaken by Bedford-Taylor [4], to overcome regularity problem for the homogeneous complex Monge-Ampère equation, they developed the theory of weak solutions and they extended the definition of intrinsic norm to a larger class of plurisubharmonic functions. Furthermore, they related it to an extremal function determined by the weak solution of the homogeneous complex Monge-Ampère equation. Among many important properties of the extremal function, they obtained the Lipschitz regularity for the solution of the homogeneous complex Monge-Ampère equation, and proved an estimate for the intrinsic norm in terms of the extremal function and the defining function of the domain. In [12], the optimal $C^{1,1}$ regularity was established for the extremal function. As a consequence, the variational characterization

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of the intrinsic norm of Bedford-Taylor is validated, along with the explicit formula for the extended norm speculated in [4].

This paper consists of two remarks related to the results in [12]. First is that the approximation of the extremal function constructed in [12] can be used to show that the extended norm of Bedford-Taylor is in fact exact the same as the Chern-Levine-Nirenberg intrinsic norm, thus it provides a proof of the original Chern-Levine-Nirenberg conjecture. The second is that the results in [12] can be generalized to any complex manifold, with the help of the existence of the plurisubharmonic function obtained in [12]. That function was used in a crucial way in [12] to get C^2 boundary estimate following an argument of Bo Guan [10]. We will use this function to establish global C^1 estimate for the homogeneous complex Monge-Ampère equation on general manifolds. We also refer Chen's work [8] on homogeneous complex Monge-Ampère equation arising from a different geometric context.

Let's recall the definitions of the Chern-Levine-Nirenberg norm [9] and the extended norm defined by Bedford-Taylor [4]. Let M be a closed complex manifold with smooth boundary $\partial M = \Gamma_1 \cup \Gamma_0$, set

$$\mathcal{F} = \{u \in C^2(M) \mid u \text{ plurisubharmonic and } 0 < u < 1 \text{ on } M\},$$

$$\mathcal{F}'_k = \{u \in \mathcal{F} \mid (dd^c u)^k = 0, \dim\gamma = 2k-1, \text{ or } du \wedge (dd^c u)^k = 0, \dim\gamma = 2k\}.$$

$\forall \gamma \in H_*(M, \mathbb{R})$ be a homology class in M,

$$N_{2k-1}\{\gamma\} = \sup_{u\in\mathcal{F}} \inf_{T\in\gamma} |T(d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim\gamma = 2k-1; \tag{1}$$

$$N_{2k}\{\gamma\} = \sup_{u\in\mathcal{F}} \inf_{T\in\gamma} |T(du \wedge d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim\gamma = 2k, \tag{2}$$

where T runs over all currents which represent γ.

It is pointed out in [9] that the intrinsic norm N_j may also be obtained as the supremum over the corresponding subclass of C^2 solutions of homogeneous complex Hessian equations in $\mathcal{F}'_k$. The most interesting case is $k = 2n-1$, elements of $\mathcal{F}'_{2n-1}$ are plurisubharmonic functions satisfying the homogeneous complex Monge-Ampère equation

$$(dd^c u)^n = 0. \tag{3}$$

In this case, associated to N_{2n-1}, there is an extremal function satisfying the Dirichlet boundary condition for the homogeneous complex Monge-Ampère equation:

$$\begin{cases} (dd^c u)^n = 0 & \text{in } M^0 \\ u|_{\Gamma_1 = 1} \\ u|_{\Gamma_0 = 0}, \end{cases} \tag{4}$$

where $d^c = i(\bar\partial - \partial)$, M^0 is the interior of M, and Γ_1 and Γ_0 are the corresponding outer and inner boundaries of M respectively.

Due to the lack of C^2 regularity for solutions of equation (4), an extended norm $\tilde{N}$ was introduced by Bedford-Taylor [4]. Set

$$\tilde{\mathcal{F}} = \{u \in C(M) | u \text{ plurisubharmonic}, \; 0 < u < 1 \text{ on } M\}.$$

$$\tilde{N}\{\gamma\} = \sup_{u\in\tilde{\mathcal{F}}} \inf_{T\in\gamma} |T(d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim\gamma = 2k-1, \tag{5}$$

$$\tilde{N}\{\gamma\} = \sup_{u\in\tilde{\mathcal{F}}} \inf_{T\in\gamma} |T(du \wedge d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim\gamma = 2k, \tag{6}$$

where the infimum this time is taken over smooth, compactly supported currents which represent γ.

$\tilde{N}$ enjoys similar properties of N, and $N \leq \tilde{N} < \infty$. They are invariants of the complex structure, and decrease under holomorphic maps.

Chern-Levine-Nirenberg observed in [9] that equation (3) also arises as the Euler equation for the functional

$$I(u) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1}. \tag{7}$$

Let

$$\mathcal{B} = \{u \in \mathcal{F} \mid u = 1 \text{ on } \Gamma_1, \, u = 0 \text{ on } \Gamma_0\}. \tag{8}$$

If $v \in \mathcal{B}$, let γ denote the $(2n-1)$-dimensional homology class of the level hypersurface $v =$ constant. Then $\forall\, T \in \gamma$, if v satisfies $(dd^c v)^n = 0$,

$$\int_T dv \wedge (dd^c v)^{n-1} = \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1} = I(v).$$

Chern-Levine-Nirenberg Conjecture [9]**:** $N\{\Gamma\} = \inf_{u\in\mathcal{B}} I(u)$.

The relationship between the intrinsic norms and the extremal function u of (4) was investigated by Bedford-Taylor [4]. They pointed out that: *if the extremal function u in* (4) *is* C^2, one has the following important representation formula,

$$\tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} \left(\frac{\partial u}{dr}\right)^n d^c r \wedge (dd^c r)^{n-1}, \tag{9}$$

where Γ_1 is the outer boundary of M and r is a defining function of Γ_1. They also observed that if the extremal function u of (4) can be approximated by functions in $\mathcal{F}_n^{'}$, then $N = \tilde{N}$ and the Chern-Levine-Nirenberg conjecture would be valid. The problem is that functions in $\mathcal{F}_n^{'}$ are C^2 plurisubharmonic functions satisfying the homogeneous complex Monge-Ampère equation. It is hard to construct such approximation due to the lack of C^2 regularity for such equation. Though in some special cases, for example on Reinhardt domains ([4]) or a perturbation of them ([1] and [16]), the extremal function is smooth.

We note that in order for equation (4) to have a plurisubharmonic solution, it is necessary that there is a plurisubharmonic subsolution v. Now suppose M is of the following form,

$$M = \bar{\Omega}^* \setminus \left(\bigcup_{j=1}^{N} \Omega_j \right), \tag{10}$$

where Ω^*, $\Omega_1, \ldots, \Omega_N$ are bounded strongly smooth pseudoconvex domains in $\mathbb{C}^n$. $\bar{\Omega}_j \subset \Omega^*$, $\forall\ j = 1, \ldots, N$, $\bar{\Omega}_1, \ldots, \bar{\Omega}_N$ are pairwise disjoint, and $\bigcup_{j=1}^{N} \Omega_j$ is holomorphic convex in Ω^*, and $\Gamma_1 = \partial\Omega^*$ and $\Gamma_0 = \bigcup_{j=1}^{N} \partial\Omega_j$. If $\Gamma = \{v = \text{constant}\}$ for some $v \in \mathcal{B}$, $\Gamma \sim \{v = 1\} \sim \{v = 0\}$ in $H_{2n-1}(M)$, the hypersurface $\{v = 1\}$ is pseudoconvex, and the hypersurface $\{v = 0\}$ is pseudoconcave. If M is embedded in $\mathbb{C}^n$, v is strictly plurisubharmonic, and M must be of the form (10). The reverse is proved in [12]: if M is of the form (10), there is $v \in PSH(M^0) \cap C^\infty(M)$

$$(dd^c V)^n > 0 \qquad \text{in } M, \tag{11}$$

such that $\Gamma_1 = \{V = 1\}$ and $\Gamma_0 = \{V = 0\}$.

The following was proved in [12].

Theorem 1. *If M is of the form* (10), *for the unique solution u of* (4), *there is a sequence $\{u_k\} \subset \mathcal{B}$ such that*

$$\|u_k\|_{C^2(M)} \le C, \quad \forall\ k, \qquad \lim_{k\to\infty} \sup (dd^c u_k)^n = 0.$$

In particular, $u \in C^{1,1}(M)$ and $\lim_{k\to\infty} \|u_k - u\|_{C^{1,\alpha}(M)} = 0, \quad \forall\ 0 < \alpha < 1$. And we have

$$\tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} \left(\frac{\partial u}{dr}\right)^n d^c r \wedge (dd^c r)^{n-1} \tag{12}$$

where r is any defining function of Ω. Moreover,

$$\tilde{N}(\{\Gamma_1\}) = \inf_{v \in \mathcal{B}} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}. \tag{13}$$

In this paper, we establish

Theorem 2. *If M is of the form* (10), *we have $N(\{\Gamma_1\}) = \tilde{N}(\{\Gamma_1\})$, and the Chern-Levine-Nirenberg conjecture is valid, that is*

$$N(\{\Gamma_1\}) = \inf_{v \in \mathcal{B}} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}. \tag{14}$$

We will work on general complex manifold M, which may not necessary to be restricted as a domain in $\mathbb{C}^n$. We assume that

M is a complex manifold, $\partial M = \Gamma_1 \bigcup \Gamma_0$ with both Γ_1 and Γ_0 are compact hypersurfaces of M, and there is $V \in \mathcal{B}$ such that $(dd^c V(z))^n > 0, \forall z \in M$. (15)

We will prove the following generalization of Theorem 1.

Theorem 3. *Suppose M is of the form* (15)*, there is a unique solution u of* (4)*, there exist a constant $C > 0$ and a sequence $\{u_k\} \subset \mathcal{B}$ such that*

$$|\Delta u_k(z)| \le C, \quad \forall\ k, \qquad \lim_{k\to\infty} \sup_{z\in M} (dd^c u_k(z))^n = 0. \tag{16}$$

In particular, $0 \le \Delta u(z) \le C$ *and* $\lim_{k\to\infty} \|u_k - u\|_{C^{1,\alpha}(M)} = 0, \quad \forall\ 0 < \alpha < 1.$ *Furthermore,*

$$N(\{\Gamma_1\}) = \tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} \left(\frac{\partial u}{dr}\right)^n d^c r \wedge (dd^c r)^{n-1} \tag{17}$$

where r is any defining function of Ω. Finally,

$$N(\{\Gamma_1\}) = \inf_{v\in\mathcal{B}} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}. \tag{18}$$

Theorem 3 implies Theorem 2. The proof Theorem 3 relies on the regularity study of equation (4). It is a degenerate elliptic fully nonlinear equation.

If M is a domain in $\mathbb{C}^n$, Caffarelli-Kohn-Nirenberg-Spruck [6] establishes $C^{1,1}$ regularity for solutions in strongly pseudoconvex domains with homogeneous boundary condition. For the Dirichlet problem (4), some pieces of the boundary are concave. In [12], we made use of the subsolution method of [10] for the second derivative estimates on the boundary (in the real case, this method was introduced by Hoffman-Rosenberg-Spruck [15] and Guan-Spruck in [11]). This type of estimates is of local feature, so the second derivative estimates on the boundary can be treated in the same way. What we will work on is the interior estimates for the degenerate complex Monge-Ampère equation on Kähler manifold. Such C^2 estimate has been established by Yau in [17]. The contribution of this paper is an interior C^1 estimate for the degenerate complex Monge-Ampère equation in general Kähler manifolds.

We remark here that the subsolution V in (15) can be guaranteed if we impose certain holomorphic convexity condition on M as one can use the pasting method developed in [12]. The subsolution V played important role in the proof boundary estimates in [12]. In this paper, the subsolution V will be crucial to prove the interior estimate. Since we are dealing the equation (4) on a general complex manifold, there may not exist a global coordinate chart. Instead, we treat equation (4) as a fully nonlinear equation on Kähler manifold (M, g), where $g = (g_{i\bar{j}}) = (V_{i\bar{j}})$ is defined by the function V in (15). We will work on the following equation with parameter $0 \le t < 1$,

$$\begin{cases} \det(g_{i\bar{j}} + \phi_{i\bar{j}}) = (1-t)\det(g_{i\bar{j}})f, \\ (g_{i\bar{j}} + \phi_{i\bar{j}}) > 0, \\ \phi|_{\Gamma_1} = 0, \\ \phi|_{\Gamma_0} = 0, \end{cases} \tag{19}$$

where f is a given positive function ($f = 1$ for (4), but we will consider general positive function f). Equation (19) is elliptic for $0 \le t < 1$. We want to prove that equation (19) has a unique smooth solution with a uniform bound on $\Delta\phi$ (independent of t). We emphasize that V is important for the C^1 estimate solutions to equation (19), and it also paves way for us to use Yau's interior C^2 estimate in [17]. We set $u = V + \phi$, where ϕ is the solution of equation (19). Therefore, u satisfies

$$\begin{cases} \det(u_{i\bar{j}}) = (1-t)f\det(V_{i\bar{j}}) \\ u|_{\Gamma_1} = 1 \\ u|_{\Gamma_0} = 0. \end{cases} \tag{20}$$

Theorem 4. *If M as in* (15), *there is a constant C depending only on M (independent of t) such that for each $0 \le t < 1$, there is a unique smooth solution u of* (19) *with*

$$|\Delta\phi(z)| \le C, \quad \forall z \in M. \tag{21}$$

We first deduce Theorem 3 from Theorem 4, following the same lines of arguments in [12].

Proof of Theorem 3. For each $0 \le t < 1$, let ϕ^t be the solution of equation (19). Set $u^t = V + \phi^t$. From Theorem 4, there is a sequence of strictly smooth plurisubharmonic functions $\{u^t\}$ satisfying (20). By (21), there is a subsequence $\{t_k\}$ that tends to 1, such that $\{u_{t_k}\}$ converges to a plurisubharmonic function u in $C^{1,\alpha}(M)$ for any $0 < \alpha < 1$. By the Convergence Theorem for complex Monge-Ampère measures (see [3]), u satisfies equation (4). Again by (21), $0 \le \Delta u \le C$.

For the sequence $\{u_k\}$, we have

$$\begin{aligned} &\int_M du_k \wedge d^c u_k \wedge (dd^c u_k)^{n-1} \\ &\quad= \int_{\Gamma_1} d^c u_k \wedge (dd^c u_k)^{n-1} - \int_M u_k (dd^c u_k)^n \\ &\quad= \int_{\Gamma_1} \left(\frac{\partial u_k}{\partial r}\right)^n d^c r \wedge (dd^c r)^{n-1} - \int_M u_k (dd^c u_k)^n \\ &\quad= \int_{\Gamma_1} \left(\frac{\partial u_k}{\partial r}\right)^n d^c r \wedge (dd^c r)^{n-1} - (1-t_k)\int_M u_k (dd^c V)^n. \end{aligned}$$

Since $u_k \to u$ in $C^{1,\alpha}(M)$, $\left(\frac{\partial u_k}{\partial r}\right)^n \to \left(\frac{\partial u}{\partial r}\right)^n$ uniformly on Γ_1. Therefore,

$$\int_{\Gamma_1} d^c u \wedge (dd^c u)^{n-1} = \int_{\Gamma_1} \left(\frac{\partial u}{\partial r}\right)^n d^c r \wedge (dd^c r)^{n-1}.$$

The proof of Theorem 3.2 in [4] yields $\tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} d^c u \wedge (dd^c u)^{n-1}$. Since $u = 1$ on Γ_1, by the Stokes Theorem,

$$\tilde{N}(\{\Gamma_1\}) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1}.$$

$\forall\ v \in \mathcal{B}$ if $v \not\equiv u$, one must have $v < u$ in M^{int}. By the Comparison Theorem,

$$\int_M du \wedge d^c u \wedge (dd^c u)^{n-1} \leq \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}.$$

That is

$$\int_M du \wedge d^c u \wedge (dd^c u)^{n-1} \leq \inf_{v \in \mathcal{B}} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}.$$

On the other hand, by the Convergent Theorem for complex Monge-Ampère measures

$$\liminf_{k \to \infty} \int_M du_k \wedge d^c u_k \wedge (dd^c u_k)^{n-1} = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1}.$$

That is,

$$\tilde{N}(\Gamma_1) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1} = \inf_{v \in \mathcal{B}} \int dv \wedge d^c v \wedge (dd^c v)^{n-1}. \tag{22}$$

Finally, if T is homological to Γ_1, there is ω such that $\partial\omega = \Gamma_1 - T$. For any $v \in \mathcal{B}$,

$$T(d^c v \wedge (dd^c v)^{n-1}) = \int_{\Gamma_1} d^c v \wedge (dd^c v)^{n-1} - \int_\omega (dd^c v)^n.$$

Applying this to u_k, we obtain

$$|T(d^c u_k \wedge (dd^c u_k)^{n-1})| \geq \int_{\Gamma_1} d^c u_k \wedge (dd^c u_k)^{n-1} - (1 - t_k) \int_M (dd^c V)^n.$$

This implies

$$N(\Gamma_1) \geq \int_{\Gamma_1} d^c u_k \wedge (dd^c u_k)^{n-1} - (1 - t_k) \int_M (dd^c V)^n.$$

Taking $k \to \infty$,

$$N(\Gamma_1) \geq \int_{\Gamma_1} d^c u \wedge (dd^c u)^{n-1} = \tilde{N}(\Gamma_1).$$

Since $\tilde{N}(\{\Gamma_1\}) \geq N(\{\Gamma_1\})$ by definition, we must have $N(\Gamma_1) = \tilde{N}(\Gamma_1)$. The Chern-Levine-Nirenberg conjecture now follows from (22). □

The rest of this paper will be devoted to the proof of Theorem 4.

Proof of Theorem 4. We show that $\forall\ 0 \leq t < 1$, $\exists!\ u_t \in C^\infty$, u_t strongly plurisubharmonic, such that u_t solves (20) and $\exists\ C > 0$, $\forall\ \ 0 \leq t < 1$

$$0 \leq \Delta u \leq C. \tag{23}$$

The uniqueness is a consequence of the comparison theorem for complex Monge-Ampère equations. In the rest of the proof, we will drop the subindex t.

We first note that since u is plurisubharmonic in M^0, and $0 \leq u \leq 1$ on ∂M, the maximum principle gives $0 \leq u(z) \leq 1\ \forall\ z \in M$. The estimate for Δu is also easy. We have $\Delta u = \Delta V + \Delta\phi = n + \Delta\phi$. Here we will make use of Yau's estimate [17]. Let $R_{i\bar{j}i\bar{j}}$ be the holomorphic bisectional curvature of the Kähler metric g, let C be a positive constant such that $C + R_{i\bar{j}i\bar{j}} \geq 2$ for all i, j. Let $\varphi = \exp(-C\phi)\Delta u$.

Lemma 1. [Yau] *There is* C_1 *depending only on* $\sup_M -\Delta f$, $\sup_M |\inf_{i\neq j} R_{i\bar{j}i\bar{j}}|$, $\sup_M f, n$, *if the maximum of* φ *is achieved at an interior point* z_0, *then*

$$\Delta u(z_0) \le C_1. \tag{24}$$

By Yau's interior C^2 estimate, we only need to get the estimates of the second derivatives of u on the boundary of M. The boundary of M consists of pieces of compact strongly pseudoconvex and pseudoconcave hypersurfaces. The second derivative estimates on strongly pseudoconvex hypersurface have been established in [6]. For general boundary under the existence of subsolution, the C^2 boundary estimate was proved by Bo Guan [10]. In [12], following the arguments in [6, 10], boundary C^2 estimates were established for M as in the form of (10), that is, M is a domain in $\mathbb{C}^n$. As these C^2 estimates are of local feature, they can be adapted to general complex manifolds without any change. Therefore, we have a uniform bound on Δu. Once Δu is bounded, the equation is uniformly elliptic and concave (for each $t < 1$). The Evans-Krylov interior and the Krylov boundary estimates can be applied here to get global $C^{2,\alpha}$ regularity (since they can be localized). In fact, with sufficient smooth boundary data, the assumption of $u \in C^{1,\gamma}$ for some $\gamma > 0$ is sufficent to get global $C^{2,\alpha}$ regularity (e.g., see Theorem 7.3 in [7]).

What is left is the gradient estimate. We will prove C^1 estimate for solution of equation (20) independent of Δu. We believe this type of estimate will be useful.

Lemma 2. *Suppose* ϕ *satisfies equation*

$$\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}})f, \tag{25}$$

where $g_{i\bar{j}} = V_{i\bar{j}}$ *for some smooth strictly plurisubharmonic function* V *and* f *is a positive function. Let* $u = V + \phi$ *and* $W = |\nabla u|^2$. *There exist constants* A *and* C_2 *depending only on* $\sup_M f^{\frac{1}{n}}, \sup_M |\nabla f^{\frac{1}{n}}|, \sup_M |V|$, $\inf_M R_{i\bar{j}i\bar{j}}$, *if the maximum of function* $H = e^{AV}W$ *is achieved at an interior point* p, *then*

$$H(p) \le C_2. \tag{26}$$

Let's first assume Lemma 2 to finish the global C^1 estimate. We only need to estimate ∇u on ∂M. Let h be the solution of

$$\begin{cases} \Delta h = 0 & \text{in } M^0 \\ h|_{\Gamma_1} = 1 \\ h|_{\Gamma_0} = 0. \end{cases} \tag{27}$$

Since $0 < \det(u_{i\bar{j}}) = (1-t)f_0 \le \det(V_{i\bar{j}})$, and

$$\Delta u > 0 = \Delta h,$$

and

$$u|_{\partial M} = V|_{\partial M} = h|_{\partial M},$$

by the Comparison Principle, $V(z) \le u(z) \le h(z)$, $\forall\ z \in M$. Therefore

$$|\nabla u(z)| \le \max\left(|\nabla V(z)|, |\nabla h(z)|\right) \le c \qquad \forall\ z \in \partial M, \tag{28}$$

i.e., $\max_{\partial M} |\nabla u| \le c$. In turn,

$$\max_M |\nabla u| \le c. \tag{29}$$

We now prove Lemma 2. Suppose the maximum of H is attained at some interior point p. We pick a holomorphic orthonormal coordinate system at the point such that $(u_{i\bar j}) = (g_{i\bar j} + \phi_{i\bar j})$ is diagonal at that point. We also have $\nabla g_{i\bar j} = \nabla g^{\alpha\bar\beta} = 0$. We may also assume that $W(p) \ge 1$.

All the calculations will be performed at p.

$$\frac{W_i}{W} + AV_i = 0, \quad \frac{W_{\bar i}}{W} + AV_{\bar i} = 0. \tag{30}$$

We have

$$\begin{aligned}
W_i &= \sum u_{\alpha i} u_{\bar\alpha} + u_\alpha u_{i\bar\alpha}, \quad W_{\bar i} = \sum u_{\alpha\bar i} u_{\bar\alpha} + u_\alpha u_{\bar i\bar\alpha},\\
W_{i\bar i} &= \sum g^{\alpha\bar\beta}_{,i\bar i} u_\alpha u_{\bar\beta} + \sum [|u_{i\alpha}|^2 + u_\alpha u_{i\bar i\bar\alpha} + u_{\bar\alpha} u_{i\bar i\alpha}] + u^2_{i\bar i},\\
|W_i|^2 &= \sum u_{\bar\alpha} u_\beta u_{i\alpha} u_{\bar i\bar\beta} + |u_i|^2 u^2_{i\bar i} + u_{\bar i} u_{i\bar i} \sum u_{\bar\alpha} u_{i\alpha} + u_i u_{i\bar i} \sum u_\alpha u_{\bar i\bar\alpha}.
\end{aligned}$$

By (30),

$$\sum u_{\bar\alpha} u_{i\alpha} = -AWV_i - u_i u_{i\bar i}, \quad \sum u_\alpha u_{\bar i\bar\alpha} = -AWV_{\bar i} - u_{\bar i} u_{i\bar i},$$

and by equation (25)

$$(\log\det(u_{i\bar j}))_\alpha = \frac{f_\alpha}{f}.$$

We have

$$|W_i|^2 = |\sum u_{\bar\alpha} u_{i\alpha}|^2 - |u_i|^2 u^2_{i\bar i} - AW u_{i\bar i}(V_i u_{\bar i} + V_{\bar i} u_i).$$

$$\begin{aligned}
0 &\ge \sum_i u^{i\bar i}\left(\frac{W_{i\bar i}}{W} - \frac{|W_i|^2}{W^2} + AV_{i\bar i}\right)\\
&= \sum u^{i\bar i}\left(\frac{g^{\alpha\bar\beta}_{,i\bar i} u_\alpha u_{\bar\beta}}{W} + AV_{i\bar i}\right)\\
&\quad + \frac{1}{W}\sum [u_\alpha (\log\det(u_{i\bar j}))_{\bar\alpha} + u_{\bar\alpha}(\log\det(u_{i\bar j}))_\alpha]\\
&\quad + \sum u^{i\bar i}\left[\left(\frac{|u_{i\alpha}|^2}{W} - \frac{|\sum u_\alpha u_{\bar i\bar\alpha}|^2}{W^2}\right) + \frac{Au_{i\bar i}(V_i u_{\bar i} + V_{\bar i} u_i)}{W}\right]\\
&\quad + \sum\left(\frac{u_{i\bar i}}{W} + \frac{|u_i|^2 u_{i\bar i}}{W^2}\right)\\
&\ge \sum u^{i\bar i}\left(\frac{g^{\alpha\bar\beta}_{,i\bar i} u_\alpha u_{\bar\beta}}{W} + AV_{i\bar i}\right) + 2\frac{1}{Wf}\left[Re\sum u_\alpha f_{\bar\alpha}\right] \qquad (31)\\
&\quad + \sum u^{i\bar i}\left[\left(\frac{|u_{i\alpha}|^2}{W} - \frac{|\sum u_\alpha u_{\bar i\bar\alpha}|^2}{W^2}\right) + 2\frac{Au_{i\bar i}}{W} Re(V_i u_{\bar i})\right].
\end{aligned}$$

Since $\frac{g^{\alpha\bar{\beta}}_{,i\bar{i}} u_\alpha u_{\bar{\beta}}}{W}$ is controlled by $\inf_M R_{i\bar{j}i\bar{j}}$, it follows from the Cauchy-Schwartz inequality,

$$0 \geq \sum u^{i\bar{i}} (\inf_M R_{k\bar{l}k\bar{l}} + AV_{i\bar{i}}) - 2\frac{\sum |u_\alpha f_\alpha|}{fW} - 2A\frac{\sum |V_i u_{\bar{i}}|}{W} \tag{32}$$

Now we may pick A sufficient large, such that

$$\inf_M R_{k\bar{l}k\bar{l}} + A \geq 1.$$

This yields

$$0 \geq \sum u^{i\bar{i}} - 2\frac{A|\nabla V| + |\nabla \log f|}{W^{\frac{1}{2}}} \geq nf^{-\frac{1}{n}} \left(1 - 2\frac{Af^{\frac{1}{n}}|\nabla V| + |\nabla f^{\frac{1}{n}}|}{W^{\frac{1}{2}}}\right). \tag{33}$$

Lemma 2 follows directly from (33). □

Added in proof

A general gradient estimate for complex Monge-Ampère equation $\det(g_{i\bar{j}} + \phi_{i\bar{j}}) = f \det(g_{i\bar{j}})$ on Kähler manifolds has been proved by Blocki in [5], also by the author in [13]. We would like to thank Phillipe Delanoe for pointing out to us that this type of gradient estimates in fact were proved by Hanani in [14] in general setting on Hermitian manifolds.

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A Radó Theorem for Locally Solvable Structures of Co-rank One

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Dedicated to Linda Rothschild

Abstract. We extend the classical theorem of Radó to locally solvable structures of co-rank one. One of the main tools in the proof is a refinement of the Baouendi-Treves approximation theorem that may be of independent interest.

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1. Introduction

A classical theorem of Radó, in the form given by Cartan, states that a continuous function defined on an open set of the complex plane which is holomorphic outside the closed set where it vanishes is holomorphic everywhere. This theorem implies easily that the same result also holds for functions of several complex variables. Radó's theorem may be regarded as a theorem about removing singularities of the Cauchy-Riemann operator, but in that theory it is customary to impose additional restrictions on the set outside which the equation holds and is wished to be removed (for instance, the set to be removed may be required to have null capacity or to have null or bounded Hausdorff measure of some dimension). The beauty of the classical result of Radó lies in the fact that the set $u^{-1}(0)$ is removed without any assumption about its size or geometric properties. The theorem was extended by replacing the set $u^{-1}(0)$ by $u^{-1}(E)$ where E is a compact subset of null analytic capacity ([St]) or is a null-set for the holomorphic Dirichlet class ([C2]). A generalization for a more general class of functions was given by Rosay and Stout

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[RS] who extended Radó's result to CR functions on strictly pseudoconvex hypersurfaces of $\mathbb{C}^n$ and other extension (in the spirit of removing singularities of CR functions) was given in [A]. For homogeneous solutions of locally solvable vector fields with smooth coefficients, a Radó type theorem was proved in [HT].

In this paper we extend Radó's theorem to homogeneous solutions of locally integrable structures of co-rank one that are locally solvable in degree one. Thus, we deal with an overdetermined system of equations

$$\begin{cases} L_1 u = 0, \\ L_2 u = 0, \\ \cdots\cdots\cdots \\ L_n u = 0, \end{cases} \tag{1.1}$$

where $L_1, \ldots, L_n$, $n \geq 1$, are pairwise commuting smooth complex vector fields defined on an open subset of $\mathbb{R}^{n+1}$ and assume that this system of vector fields has local first integrals at every point and it is solvable in the sense that the equation

$$\begin{cases} L_1 u = f_1, \\ L_2 u = f_2, \\ \cdots\cdots\cdots \\ L_n u = f_n, \end{cases} \tag{1.2}$$

can be locally solved for all smooth right-hand sides that satisfy the compatibility conditions $L_j f_k = L_k f_j$, $1 \leq j, k \leq n$ (see Section 3 for precise statements). Under these condition it is shown that if u is continuous and satisfies (1.1) outside $u^{-1}(0)$ then it satisfies (1.1) everywhere (Theorem 4.1). This solvability hypothesis can be characterized in terms of the connectedness properties of the fibers of local first integrals ([CT], [CH]) and this characterization is one of the main ingredients in the proof of Theorem 4.1, which is given in Section 4. Another key tool is a refinement of the Baouendi-Treves approximation theorem, which seems to have interest *per se* and it is stated and proved in Section 2 for general locally integrable structures. In Section 5 we apply Theorem 4.1 to obtain a result on uniqueness in the Cauchy problem for continuous solutions with Cauchy data on rough initial surfaces.

2. The approximation theorem

The approximation formula ([BT1], [BT2], [T1], [T2], [BCH]) is of local nature and we will restrict our attention to a locally integrable structure $\mathcal{L}$ defined in an open subset Ω of $\mathbb{R}^N$ over which $\mathcal{L}^\perp$ is spanned by the differentials $dZ_1, \ldots, dZ_m$ of m smooth functions $Z_j \in C^\infty(\Omega)$, $j = 1, \ldots, m$, at every point of Ω. Thus, if n is the rank of $\mathcal{L}$, we recall that $N = n + m$.

Given a continuous function $u \in C(\Omega)$ we say that u is a homogeneous solution of $\mathcal{L}$ and write $\mathcal{L}u = 0$ if, for every local section L of $\mathcal{L}$ defined on an open subset $U \subset \Omega$,

$$Lu = 0 \quad \text{on } U \text{ in the sense of distributions.}$$

Simple examples of homogeneous solutions of $\mathcal{L}$ are the constant functions and also the functions $Z_1, \dots, Z_m$, since $LZ_j = \langle dZ_j, L\rangle = 0$ because $dZ_j \in \mathcal{L}^\perp$, $j = 1, \dots, m$. By the Leibniz rule, any product of smooth homogeneous solutions is again a homogeneous solution, so a polynomial with constant coefficients in the m functions Z_j, i.e., a function of the form

$$P(Z) = \sum_{|\alpha|\le d} c_\alpha Z^\alpha, \quad \alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m, \quad c_\alpha \in \mathbb{C}, \tag{2.1}$$

is also a homogeneous solution. The classical approximation theorem for continuous functions states that any continuous solution u of $\mathcal{L}u = 0$ can be uniformly approximated by polynomial solutions such as (2.1). More precisely:

Theorem 2.1. *Let $\mathcal{L}$ be a locally integrable structure on Ω and assume that $dZ_1, \dots, dZ_m$ span $\mathcal{L}^\perp$ at every point of Ω. Then, for any $p \in \Omega$, there exist two open sets U and W, with $p \in U \subset \overline{U} \subset W \subset \Omega$, such that*

every $u \in C(W)$ that satisfies $\mathcal{L}u = 0$ on W is the uniform limit of a sequence of polynomial solutions $P_j(Z_1, \dots, Z_m)$:

$$u = \lim_{j\to\infty} P_j \circ Z \quad \textit{uniformly in } \overline{U}.$$

In this section we will prove a refinement of the approximation theorem (Theorem 2.2) that we will later use in the proof of the paper's main result. In order to prove this variation, we start by reviewing the main steps in the proof of the classical approximation theorem. The first one is to choose local coordinates

$$\{x_1, \dots, x_m, t_1, \dots, t_n\}$$

defined on a neighborhood of the point p and vanishing at p so that, for some smooth, real-valued functions $\varphi_1, \dots, \varphi_m$ defined on a neighborhood of the origin and satisfying

$$\varphi_k(0,0) = 0, \quad d_x\varphi_k(0,0) = 0, \quad k = 1, \dots, m,$$

the functions Z_k, $k = 1, \dots, m$, may be written as

$$Z_k(x,t) = x_k + i\varphi_k(x,t), \quad k = 1, \dots, m, \tag{2.2}$$

on a neighborhood of the origin. Then we choose $R > 0$ such that if

$$V = \{q : \quad |x(q)| < R,\ |t(q)| < R\}$$

then, on a neighborhood of $\overline{V}$ we have

$$\left\| \left(\frac{\partial\varphi_j(x,t)}{\partial x_k} \right) \right\| < \frac{1}{2}, \quad (x,t) \in \overline{V}, \tag{2.3}$$

where the double bar indicates the norm of the matrix $\varphi_x(x,t) = (\partial\varphi_j(x,t)/\partial x_k)$ as a linear operator in $\mathbb{R}^m$. Modifying the functions φ_k's off a neighborhood of $\overline{V}$ we may assume without loss of generality that the functions $\varphi_k(x,t)$, $k = 1, \dots, m$, are defined throughout $\mathbb{R}^N$, have compact support and satisfy (2.3) everywhere, that is

$$\left\| \left(\frac{\partial \varphi_j(x,t)}{\partial x_k} \right) \right\| < \frac{1}{2}, \quad (x,t) \in \mathbb{R}^N. \tag{2.3$'$}$$

Modifying also $\mathcal{L}$ off a neighborhood of $\overline{V}$ we may assume as well that the differentials dZ_j, $j = 1, \ldots, m$, given by (2.2), span $\mathcal{L}^\perp$ over $\mathbb{R}^N$. Of course, the new structure $\mathcal{L}$ and the old one coincide on V so any conclusion we draw about the new $\mathcal{L}$ on V will hold as well for the original $\mathcal{L}$. The vector fields

$$M_k = \sum_{\ell=1}^m \mu_{k\ell}(x,t) \frac{\partial}{\partial x_\ell}, \quad k = 1, \ldots, m,$$

characterized by the relations

$$M_k Z_\ell = \delta_{k\ell} \quad k, \ell = 1, \ldots, m,$$

and the vectors fields

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^m \frac{\partial \varphi_k}{\partial t_j}(x,t) M_k, \quad j = 1, \ldots, n,$$

are linearly independent and satisfy $L_j Z_k = 0$, for $j = 1, \ldots, n$, $k = 1, \ldots, m$. Hence, $L_1, \ldots, L_n$ span $\mathcal{L}$ at every point while the $N = n + m$ vector fields

$$L_1, \ldots, L_n, M_1 \ldots, M_m,$$

are pairwise commuting and span $\mathbb{C}T_p(\mathbb{R}^N)$, $p \in \mathbb{R}^N$. Since

$$dZ_1, \ldots, dZ_m, dt_1, \ldots, dt_n \quad \text{span } \mathbb{C}T^*\mathbb{R}^N$$

the differential dw of a C^1 function $w(x,t)$ may be expressed in this basis. In fact, we have

$$dw = \sum_{j=1}^n L_j w \, dt_j + \sum_{k=1}^m M_k w \, dZ_k$$

which may be checked by observing that $L_j Z_k = 0$ and $M_k t_j = 0$ for $1 \leq j \leq n$ and $1 \leq k \leq m$, while $L_j t_k = \delta_{jk}$ for $1 \leq j, k \leq n$ and $M_k Z_j = \delta_{jk}$ for $1 \leq j, k \leq m$ ($\delta_{jk} =$ Kronecker delta). At this point, the open set W in the statement of Theorem 2.1 is chosen as any fixed neighborhood of $\overline{V}$ in Ω. That $u \in C(W)$ satisfies $\mathcal{L}u = 0$ is equivalent to saying that it satisfies on W the overdetermined system of equations

$$\begin{cases} L_1 u = 0, \\ L_2 u = 0, \\ \cdots\cdots\cdots \\ L_n u = 0. \end{cases} \tag{2.4}$$

Given such u we define a family of functions $\{E_\tau u\}$ that depend on a real parameter τ, $0 < \tau < \infty$, by means of the formula

$$E_\tau u(x,t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau[Z(x,t) - Z(x',0)]^2} u(x',0) h(x') \det Z_x(x',0) \, dx'$$

which we now discuss. For $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{C}^m$ we will use the notation $[\zeta]^2 = \zeta_1^2 + \cdots + \zeta_m^2$, which explains the meaning of $[Z(x,t) - Z(x',0)]^2$ in the formula. The function $h(x) \in C_c^\infty(\mathbb{R}^m)$ satisfies $h(x) = 0$ for $|x| \geq R$ and $h(x) = 1$ in a neighborhood of $|x| \leq R/2$ (R was defined right before (2.3)). Since u is assumed to be defined in a neighborhood of $\overline{V}$, the product $u(x',0)h(x')$ is well defined on $\mathbb{R}^m$, compactly supported and continuous. Furthermore, since the exponential in the integrand is an entire function of $(Z_1, \dots, Z_m)$, the chain rule shows that it satisfies the homogenous system of equations (2.4) and the same holds for $E_\tau u(x,t)$ by differentiation under the integral sign. Then Theorem 2.1 is proved by showing that $E_\tau u(x,t) \to u(x,t)$ as $\tau \to \infty$ uniformly for $|x| < R/4$ and $|t| < T < R$ if T is conveniently small. In particular, the set U in the statement of Theorem 2.1 may be taken as

$$\begin{aligned} U &= B_1 \times B_2, \\ B_1 &= \{x \in \mathbb{R}^m : \quad |x| < R/4\}, \\ B_2 &= \{t \in \mathbb{R}^n : \quad |t| < T\}. \end{aligned}$$

Once this is proved, approximating the exponential $e^{-\tau[\zeta]^2}$ (for fixed large τ) by the partial sum of degree k, $P_k(\zeta)$, of its Taylor series on a fixed polydisk that contains the set $\{\sqrt{\tau}(Z(x,t) - Z(x',0)) : |x|, |x'| < R, |t| < R\}$, a sequence of polynomials in $Z(x,t)$ that approximate uniformly $E_\tau u(x,t)$ for $|x| < R/4$ and $|t| < T$ as $k \to \infty$ is easily constructed.

Thus, the main task is to prove that $E_\tau u(x,t) \to u(x,t)$ as $\tau \to \infty$ uniformly on $B_1 \times B_2$ and in order to do that one proves first that a convenient modification of the operator E_τ, to wit,

$$G_\tau u(x,t) = (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau[Z(x,t) - Z(x',t)]^2} u(x',t) h(x') \det Z_x(x',t)\, dx',$$

converges uniformly to u on $B_1 \times B_2$ as $\tau \to \infty$. This is easy because (2.3′) ensures that the operator $u \mapsto G_\tau u$ is very close to the convolution of u with a Gaussian, which is a well-known approximation of the identity. In particular, the uniform convergence $G_\tau \to u$ holds for any continuous u and it is irrelevant at this point whether u satisfies the equation $\mathcal{L}u = 0$ or not. After $G_\tau u \to u$ has been proved, it remains to estimate the difference $R_\tau u = G_\tau u - E_\tau u$ and it is here that the fact that $\mathcal{L}u = 0$ is crucial.

Let B_1 and B_2 be as described before in the outline of the proof of Theorem 2.1 and set $\tilde{B}_1 = \{x \in \mathbb{R}^m : \quad |x| < R\}$. In the version we want to prove u need not be a solution in a neighborhood of $\overline{V}$ but on a smaller open subset of $\tilde{B}_1 \times B_2$. With this notation we have

Theorem 2.2. *Let u be continuous on $\overline{\tilde{B}_1} \times \overline{B_2}$ and assume that there is an open and connected set ω, $0 \in \omega \subset B_2$, such that*

$$\mathcal{L}u = 0, \quad \text{on } \tilde{B}_1 \times \omega.$$

Then $E_\tau u(x,t) \to u(x,t)$ uniformly on compact subsets of $B_1 \times \omega$ as $\tau \to \infty$.

Remark 2.3. The main point in Theorem 2.2 is that as soon as the equation holds on $\tilde{B}_1 \times \omega$, in order to obtain a set where the approximation holds we do not need to replace ω by a smaller subset, although we must shrink $\tilde{B}_1$ to B_1 and the radius T of B_2 has been initially taken small as compared to the radius R of $\tilde{B}_1$. Under some more restrictive circumstances, we may even avoid taking T is small, as the proof of Theorem 4.1 below shows.

Proof. The formula that defines $E_\tau u(x,t)$ only takes into account the values of $u(x,0)$. It will be enough to prove that $E_\tau u \to u$ uniformly on compact subsets of $B_1 \times \omega$ as $j \to \infty$. The argument that shows in the classical setup that $G_\tau u \to u$ uniformly on $\tilde{B}_1 \times B_2$ applies here word by word, because it only uses the fact that u is continuous on the closure of $\tilde{B}_1 \times B_2$ and it is carried out by freezing $t \in B_2$ and showing that $G_\tau u$ is an approximation of the identity on $\mathbb{R}^m$, uniformly in $t \in B_2$. Hence, the proof is reduced to showing that $R_\tau u = G_\tau u - E_\tau u$ converges uniformly to 0 on compact subsets of $B_1 \times \omega$.

When u satisfies $\mathcal{L}u = 0$ throughout $\tilde{B}_1 \times B_2$, we have the formula

$$R_\tau u(x,t) = \int_{[0,t]} \sum_{j=1}^{n} r_j(x,t,t',\tau)\, dt'_j, \tag{2.5}$$

where

$$r_j(x,t,t',\tau) = \\ (\tau/\pi)^{m/2} \int_{\mathbb{R}^m} e^{-\tau[Z(x,t)-Z(x',t')]^2} u(x',t') L_j h(x',t') \det Z_x(x',t')\, dx' \tag{2.6}$$

and $[0,t]$ denotes the straight segment joining 0 to t. This may be shown by writing for fixed ζ and τ

$$g(t') = \tilde{G}_\tau u(\zeta,t') = \int_{\mathbb{R}^m} e^{-\tau[\zeta - Z(x',t')]^2} u(x',t') h(x') \det Z_x(x',t')\, dx'$$

and applying the fundamental theorem of calculus

$$g(t) - g(0) = \int_{[0,t]} \sum_{j=1}^{n} \frac{\partial g}{\partial t'_j}(t')\, dt'_j. \tag{2.7}$$

Then a computation that exploits that $L_j u = 0$, $j = 1, \ldots, n$, shows that (see [BCH, p. 64] for details)

$$\frac{\partial g}{\partial t'_j}(t') = \tilde{r}_j(\zeta,t',\tau) \tag{2.8}$$

where

$$\tilde{r}_j(\zeta,t',\tau) = \int_{\mathbb{R}^m} e^{-\tau[\zeta - Z(x',t')]^2} u(x',t') L_j h(x',t') \det Z_x(x',t')\, dx'.$$

Hence, (2.7) for $\zeta = Z(x,t)$ shows that $R_\tau u = G_\tau u - E_\tau u$ is given by (2.5) and (2.6).

Let's return to the case in which u is only known to satisfy $L_j u = 0$, $j = 1, \ldots, n$, on $\tilde{B}_1 \times \omega$. We still have, for $t \in \omega$,

$$g(t) - g(0) = \int_{\gamma_t} \sum_{j=1}^{n} \frac{\partial g}{\partial t'_j}(t')\, dt'_j. \tag{2.7$'$}$$

where γ_t denotes a polygonal path contained in ω that joins the origin to t. On the other hand, (2.8) remains valid in the new situation. This is true because its proof depends on integration by parts with respect to x – which can be performed as well on $\tilde{B}_1 \times \omega$ – and on local arguments. Thus, we get

$$R_\tau u(x,t) = \int_{\gamma_t} \sum_{j=1}^{n} r_j(x,t,t',\tau)\, dt'_j, \quad (x,t) \in \tilde{B}_1 \times \omega. \tag{2.5$'$}$$

This gives the estimate

$$|R_\tau u(x,t)| \le C|\gamma_t| \max_{1 \le j \le n} \sup_{t' \in \omega} |r_j(x,t,t',\tau)|, \quad (x,t) \in B_1 \times \omega.$$

However, due to the fact that the factor $L_j h(x',t')$ vanishes for $|x'| \ge R/2$, we have

$$\left| e^{-\tau[Z(x,t) - Z(x',t')]^2} \right| \le e^{-c\tau}, \quad (x,t) \in B_1 \times B_2,\ |x'| \ge R/2,\ t' \in B_2,$$

for some $c > 0$. This follows, taking account of (2.3$'$), from

$$\begin{aligned} \Re[Z(x,t) - Z(x',t')]^2 &\ge |x - x'|^2 - |\varphi(x,t) - \varphi(x',t')|^2 \\ &\ge \frac{|x-x'|^2}{2} - |\varphi(x',t) - \varphi(x',t')|^2 \\ &\ge c. \end{aligned} \tag{2.9}$$

Note that $|x - x'| \ge R/4$ for $|x'| \ge R/2$ and $|x| \le R/4$, while the term $|\varphi(x',t) - \varphi(x',t')| \le C|t - t'|$ will be small if t and t' are both small which may be obtained by taking T small. Thus

$$|R_\tau u(x,t)| \le C|\gamma_t| e^{-c\tau}, \quad (x,t) \in B_1 \times \omega. \tag{2.10}$$

If $K \subset\subset \omega$, there is a constant C_K such that any $t \in K$ can be reached from the origin by a polygonal line of length bounded by C_K so (2.10) shows that $|R_\tau u(x,t)| \to 0$ uniformly on $B_1 \times K$. □

Corollary 2.4. *Under the hypotheses of the theorem, there is a sequence of polynomial solutions $P_j(Z_1, \ldots, Z_m)$ that converges uniformly to u on compact subsets of $B_1 \times \omega$ as $j \to \infty$.* □

3. Structures of co-rank one

A smooth locally integrable structure $\mathcal{L}$ of rank $n \geq 1$ defined on an open subset $\Omega \subset \mathbb{R}^{n+1}$ is said to be a structure of co-rank one. Thus $\mathcal{L}^{\perp}$ is locally spanned by a single function Z that, in appropriate local coordinates $(x, t_1, \ldots, t_n)$ centered around a given point, may be written as

$$Z(x,t) = x + i\varphi(x,t), \quad |x| < a, \quad |t| < r,$$

where $\varphi(x,t)$ is smooth, real valued and satisfies $\varphi(0,0) = \varphi_x(0,0) = 0$. Then, $\mathcal{L}$ is locally spanned by the vector fields

$$L_j = \frac{\partial}{\partial t_j} - i\frac{\varphi_{t_j}}{1+i\varphi_x}\frac{\partial}{\partial x}, \quad j = 1, \ldots, n,$$

on $X \doteq (-a,a) \times \{|t| < r\}$. It turns out that $[L_j, L_k] = 0$, $1 \leq j,k \leq n$. Given an open set $Y \subset X$, consider the space of p-forms

$$C^{\infty}(Y, \textstyle\bigwedge^p) \doteq \Big\{ u = \displaystyle\sum_{|J|=p} u_J(x,t) dt_J, \ u_J \in C^{\infty}(Y) \Big\}$$

as well as the differential complex

$$L : C^{\infty}(Y, \textstyle\bigwedge^p) \longrightarrow C^{\infty}(Y, \textstyle\bigwedge^{p+1})$$

defined by

$$L = \sum_{|J|=p} \sum_{j=1}^{n} L_j u_J(x,t)\, dt_j \wedge dt_J.$$

The fact that $L^2 = 0$ ensues from the relations $[L_j, L_k] = 0$, $1 \leq j,k \leq n$.

Definition 3.1. The operator L is said to be solvable at $\omega_0 \in \Omega$ in degree q, $1 \leq q \leq n$, if for every open neighborhood $Y \subset X$ of ω_0 and $f \in C^{\infty}(Y, \bigwedge^q)$ such that $Lf = 0$, there exists an open neighborhood $Y' \subset Y$ of ω_0 and a $(q-1)$-form $u \in C^{\infty}(Y', \bigwedge^{q-1})$ such that $Lu = f$ in Y'. If this holds for every $\omega_0 \in \Omega$ we say that the structure $\mathcal{L}$ is locally solvable in Ω in degree q.

If $z_0 = x_0 + iy_0 \in \mathbb{C}$ and $Y \subset X$ we will refer to the set

$$\mathcal{F}(z_0, Y) = \{(x,t) \in Y : \ Z(x,t) = z_0\} = Z^{-1}(z_0) \cap Y$$

as *the fiber of the map* $Z : X \longrightarrow \mathbb{C}$ *over* Y. The local solvability in degree q can be characterized in terms of the homology of the fibers of Z for any degree $1 \leq q \leq n$, as it was conjectured by Treves in [T3]. The full solution of this conjecture took several years (see [CH] and the references therein). In this paper we will only consider locally solvable structures of co-rank one that are locally solvable in degree $q = 1$. For $q = 1$, the geometric characterization of local solvability at the origin means that ([CT],[CH],[MT]), given any open neighborhood X of the origin there is another open neighborhood $Y \subset X$ of the origin such that, for every regular value $z_0 \in \mathbb{C}$ of $Z : X \longrightarrow \mathbb{C}$, either $\mathcal{F}(z_0, Y)$ is empty or else the homomorphism

$$\tilde{H}_0(\mathcal{F}(z_0, Y), \mathbb{C}) \longrightarrow \tilde{H}_0(\mathcal{F}(z_0, X), \mathbb{C}) \qquad (*)_0$$

induced by the inclusion map $\mathcal{F}(z_0, Y) \subset \mathcal{F}(z_0, X)$ is identically zero. We are denoting by $\tilde{H}_0(M, \mathbb{C})$ the 0th reduced singular homology space of M with complex coefficients. In other words, $(*)_0$ means that there is at most one connected component C_q of $\mathcal{F}(z_0, X)$ that intersects Y. Thus, if $q \in Y$, $z_0 = Z(q)$ is a regular value and C_q is the connected component of $\mathcal{F}(z_0, X)$ that contains q, it follows that

$$\mathcal{F}(z_0, Y) = Y \cap \mathcal{F}(z_0, X) = Y \cap C_q.$$

4. A Radó theorem for structures of co-rank one

Consider a smooth locally integrable structure $\mathcal{L}$ of rank $n \geq 1$ defined on an open subset $\Omega \subset \mathbb{R}^{n+1}$. A function u defined on Ω is a Radó function if

i) $u \in C(\Omega)$ and
ii) satisfies the differential equation $\mathcal{L}u = 0$ on $\Omega \setminus u^{-1}(0)$, in the weak sense.

We say that $\mathcal{L}$ has the Radó property if every Radó function is a homogeneous solution on Ω, i.e., the singular set where u vanishes and where the equation is a priori not satisfied can be removed and the equation $\mathcal{L}u = 0$ holds everywhere.

Theorem 4.1. *Assume that $\mathcal{L}$ is locally solvable in degree 1 in Ω. Then $\mathcal{L}$ has the Radó property.*

The Radó property has a local nature: it is enough to show that a Radó function u satisfies the equation $\mathcal{L}u = 0$ in a neighborhood of an arbitrary point $p \in \Omega$ such that $u(p) = 0$. We may choose local coordinates $x, t_1, \dots, t_n$ such that $x(p) = t_j(p) = 0$, $j = 1, \dots, n$, in which a first integral $Z(x,t)$ has the form

$$Z(x,t) = x + i\varphi(x,t), \quad |x| \leq 1, \quad |t| = |(t_1, \dots, t_n)| \leq 1,$$

where φ is real valued and $\varphi(0,0) = \varphi_x(0,0) = 0$. Let us write

$$I = (-1,1), \quad B = \{t : \ |t| < 1\}, \quad X = I \times B.$$

By Theorem 2.1, we may further assume without loss of generality that any continuous solution of $\mathcal{L}v = 0$ defined on a neighborhood of $|x| \leq 1$, $|t| \leq 1$, is uniformly approximated by $E_\tau v$ for $|x| < a$, $|t| < T$, where a and T are convenient positive small values. Of course, we cannot apply this to our Radó function u since u is not known to satisfy the equation everywhere.

In order to prove that $\mathcal{L}u = 0$ in some neighborhood of the origin we will consider different cases.

Case 1

Assume that $\nabla_t \varphi(0,0) \neq 0$, say $\partial\varphi(0,0)/\partial t_1 \neq 0$ (this is the simple elliptic case). Then, replacing the coordinate function t_1 by φ and leaving $t_2, \dots, t_n$ unchanged, we obtain a local change of coordinates defined in a small ball centered at the origin. In the new coordinates we have $\varphi(x,t) \equiv t_1$. Then the system (2.4) becomes

$$\begin{cases} \left(\dfrac{\partial}{\partial t_1} - i\dfrac{\partial}{\partial x}\right) u = 0, \\ \dfrac{\partial}{\partial t_2} u = 0, \\ \cdots\cdots\cdots\cdots \\ \dfrac{\partial}{\partial t_n} u = 0. \end{cases}$$

Take an arbitrary point $p = (x_0, \tau_1, \tau')$, $\tau' = (\tau_2, \ldots, \tau_n)$ such that $u(p) \neq 0$ and a cube Q centered at p that does not intersect the zero set of u. Choosing Q sufficiently small, we may approximate u uniformly on Q by polynomials in the first integral $Z = x+it_1$. Then, for fixed t', the restricted function $u_{t'}(t_1, x) = u(x, t_1, t')$ is a holomorphic function of $x + it_1$ on a slice of Q. Keeping $\tau' = t'_0$ fixed and varying (x_0, τ_1), it turns out that $u_{t'_0}(t_1, x) = u(x, t_1, t'_0)$ is a holomorphic function of $x + it_1$ outside its zero set so, by the classical Radó theorem it is a holomorphic everywhere. Similarly, keeping (x_0, τ_1) fixed and letting t' vary, we see that the function $t' \mapsto u(x_0, \tau_1, t')$ is locally constant on the set $\{t' : \ u(x_0, \tau_1, t') \neq 0\}$, thus constant on its connected components. The continuity of u then shows that $t' \mapsto u(x_0, \tau_1, t')$ is constant for fixed (x_0, τ_1). Hence, u is independent of $t' = (t_2, \ldots, t_n)$ and the restricted function $u_{t'_0}(t_1, x) = u(x, t_1, t'_0)$ is a holomorphic function of $x + it_1$ so u satisfies $\mathcal{L}u = 0$ in a neighborhood of the origin.

We already know that $\varphi_x(0,0) = 0$ and in view of Case 1 we will assume from now on that

$$\nabla_{x,t}\varphi(0,0) = 0. \tag{4.1}$$

We recall that, for $z_0 = x_0 + iy_0 \in \mathbb{C}$ and $Y \subset X$, the set

$$\mathcal{F}(z_0, Y) = \{(x,t) \in Y : \ Z(x,t) = z_0\} = Z^{-1}(z_0) \cap Y$$

is referred to as *the fiber of the map* $Z : X \longrightarrow \mathbb{C}$ *over* Y. To deal with the next cases, the following lemma will be important; its proof is a consequence of the classical approximation theorem.

Lemma 4.2. *The Radó function u is constant on the connected components of the fibers of Z over X.*

Proof. Let $q \in X$. Assume first that $u(q) \neq 0$. Then u is a homogeneous solution of $\mathcal{L}$ on a neighborhood of q and by a standard consequence of Theorem 2.1 applied at the point q, u must be constant on the fibers of some first integral Z_1 of $\mathcal{L}$ defined on a sufficiently small neighborhood of q. The germs of the fibers at q are invariant objects attached to $\mathcal{L}$ and do not depend on the particular first integral, i.e., replacing Z_1 by Z, there exists a neighborhood W of q such that u is constant on $\mathcal{F}(z_0, W)$, $z_0 = Z(q) = x_0 + iy_0$. Let $\{x_0\} \times \mathcal{C}_q$ be the connected component of $\mathcal{F}(z_0, X) \setminus u^{-1}(0)$ that contains q and denote by $\{x_0\} \times \mathcal{C}_q^{\#}$ the connected component of $\mathcal{F}(z_0, X)$ that contains q, so $\mathcal{C}_q \subset \mathcal{C}_q^{\#}$. We have seen that u is locally constant on $\{x_0\} \times \mathcal{C}_q$, hence it assumes a constant value $c \neq 0$ on $\{x_0\} \times \mathcal{C}_q$. This implies that u cannot vanish at any point on the closure of $\{x_0\} \times \mathcal{C}_q$ in X and therefore $\mathcal{C}_q$ is both open and closed in $\mathcal{C}_q^{\#}$ so $\mathcal{C}_q = \mathcal{C}_q^{\#}$. Hence, $\{x_0\} \times \mathcal{C}_q$ is a connected component of $\mathcal{F}(z_0, X)$ and the lemma is proved in this case.

Assume now $u(q) = 0$ and let $\{x_0\} \times \mathcal{C}_q$ be the connected component of $\mathcal{F}(z_0, X)$ that contains q. We will show that $\{x_0\} \times \mathcal{C}_q \subset u^{-1}(0)$. Suppose there exists a point $q_1 \in \{x_0\} \times \mathcal{C}_q \setminus u^{-1}(0) \neq \emptyset$. By our previous reasoning, u would assume a constant value $c \neq 0$ on the connected component $\{x_0\} \times \mathcal{C}_{q_1}$ of $\mathcal{F}(z_0, X)$ that contains q_1. Since $q_1 \in \{x_0\} \times (\mathcal{C}_{q_1} \cap \mathcal{C}_q)$ we should have $\mathcal{C}_{q_1} = \mathcal{C}_q$ and consequently $u(q) = c \neq 0$, a contradiction. Hence, u vanishes identically on $\mathcal{F}(z_0, X)$. □

At this point we will exploit the assumption that $\mathcal{L}$ is locally solvable in degree one. By the geometric characterization of locally solvable structures of corank one, we know that given any open neighborhood X of the origin there is another open neighborhood $Y \subset X$ of the origin such that, for every regular value $z_0 \in \mathbb{C}$ of $Z : X \longrightarrow \mathbb{C}$, either $\mathcal{F}(z_0, Y)$ is empty or else the homomorphism

$$\tilde{H}_0(\mathcal{F}(z_0, Y), \mathbb{C}) \longrightarrow \tilde{H}_0(\mathcal{F}(z_0, X), \mathbb{C}) \qquad (*)_0$$

induced by the inclusion map $\mathcal{F}(z_0, Y) \subset \mathcal{F}(z_0, X)$ is identically zero. As mentioned at the end of Section 3, this implies that if $q \in Y$, $z_0 = Z(q)$ is a regular value and C_q is the connected component of $\mathcal{F}(z_0, X)$ that contains q, it follows that

$$\mathcal{F}(z_0, Y) = Y \cap \mathcal{F}(z_0, X) = Y \cap C_q.$$

By Lemma 4.2 u is constant on C_q. This shows that u is constant on the regular fibers $\mathcal{F}(z_0, Y)$ of Z over Y but, using Sard's theorem, a continuity argument shows that u is constant on all fibers $\mathcal{F}(z_0, Y)$, whether regular or not. Hence, after restricting u to $\overline{Y}$, we may write

$$u(x,t) = U \circ Z(x,t), \quad (x,t) \in \overline{Y}, \qquad (4.2)$$

with $U \in C^0(Z(\overline{Y}))$. Once $U(x, y)$ has been defined, (4.2) will still hold if we replace Y by a smaller neighborhood of the origin. Thus, redefining I and B as $I = (-a, a)$, $B = \{|t| \leq T\}$, with $0 < a < 1$, $0 < T < 1$ conveniently small, we may assume from the start that

$$u(x,t) = U \circ Z(x,t), \quad (x,t) \in \overline{X}.$$

Shrinking I and B further if necessary we may also assume, recalling (4.1), that

$$|\nabla_{x,t}\varphi(x,t)| \leq \frac{1}{24} \quad \text{for } (x,t) \in \overline{X}. \qquad (4.3)$$

Write

$$M(x) = \sup_{|t| \leq T} \varphi(x,t), \qquad m(x) = \inf_{|t| \leq T} \varphi(x,t)$$

so

$$Z([-a,a] \times \overline{B}) = \big\{x + iy : \quad |x| \leq a,\ m(x) \leq y \leq M(x)\big\}.$$

The functions $M(x)$ and $m(x)$ are continuous and we may write

$$\{x \in (-a, a) : \quad m(x) < M(x)\} = \bigcup_{j=1}^{N} I_j$$

where $I_j \subset (-a, a)$ is an open interval for $1 \leq j \leq N \leq \infty$. We also write $\mathcal{N} = (-a, a) \setminus \bigcup_{j=1}^{N} I_j$ and

$$D_j = \{x + iy : \quad x \in I_j, \ m(x) < y < M(x)\}, \quad 1 \leq j \leq N.$$

Case 2

We suppose now that $0 = Z(0, 0) \in \bigcup_j D_j$, i.e., we will assume that $0 \in I_j$ for some j (that we may take as $j = 1$) and $m(0) < 0 = \varphi(0, 0) < M(0)$. If $(0, 0)$ is in the interior of $u^{-1}(0)$, it is clear that $\mathcal{L}u$ vanishes in a neighborhood of $(0, 0)$ as we wish to prove, so we may assume that u does not vanish identically in any neighborhood of $(0, 0)$. Hence, there are points $q = (x_0, t_0) \in I_1 \times B$ such that $z_0 \doteq Z(q) = x_0 + i\varphi(x_0, t_0) \doteq x_0 + iy_0 \in D_1$ and $u(q) \neq 0$. Let $\{x_0\} \times \mathcal{C}_q$ be the connected component of $\mathcal{F}(z_0, I_1 \times B)$ that contains q. Cover $\overline{\mathcal{C}_q}$ with a finite number L of balls of radius $\delta > 0$ centered at points of $\mathcal{C}_q$ and call ω_δ the union of these balls. Notice that any two points in ω_δ can be joined by a polygonal line γ of length $|\gamma| \leq (L + 2)\delta$. Thus, ω_δ is a connected open set that contains $\mathcal{C}_q$ and, since $m(x_0) < y_0 < M(x_0)$, there is no restriction in assuming that q has been chosen so that $\varphi(x_0, t)$ assumes on ω_δ some values which are larger than y_0 as well as some values which are smaller than y_0. Indeed, consider a smooth curve $\gamma(s) : [0, 1] \longrightarrow \overline{B}$ such that for some $0 < s_1 < 1$, $\varphi(x_0, \gamma(0)) = m(x_0)$, $\varphi(x_0, \gamma(s_1)) = y_0$, $\varphi(x_0, \gamma(1)) = M(x_0)$ and $\varphi(x_0, \gamma(s)) \leq y_0$ for any $0 \leq s \leq s_1$. Having fixed $\gamma(s)$, we may choose a largest $s_1 \in (0, 1)$ with that property. This means that there are points $s > s_1$ arbitrarily close to s_1 such that $\varphi(x_0, \gamma(s)) > y_0$. Let $[s_0, s_1]$ be the connected component of

$$\{s \in [0, 1] : \ \varphi(x_0, \gamma(s)) = y_0\}$$

that contains s_1 (note that $0 < s_0 \leq s_1 < 1$). Set $q = (x_0, \gamma(s_0))$, $q' = (x_0, \gamma(s_1))$ and notice that $\gamma(s) \in \mathcal{C}_q$ for $s_0 \leq s \leq s_1$ by connectedness so $q' \in \mathcal{C}_q$. For any $\varepsilon > 0$ there exist $\varepsilon_0, \varepsilon_1 \in (0, \varepsilon)$ such that $\gamma(s_0 - \varepsilon_0)$ and $\gamma(s_1 + \varepsilon_1)$ are points in ω_δ on which $\varphi(x_0, t)$ takes values respectively smaller and larger than y_0.

Next, for small $\delta > 0$, we consider the approximation operator $E_\tau u$ on $(x_0 - \delta, x_0 + \delta) \times B \doteq I_\delta \times B$ (with initial trace taken at $t = t_0$) and we wish to prove that $E_\tau u$ converges to u uniformly on $I_\delta \times \omega_\delta$, after choosing $\delta > 0$ sufficiently small to ensure that u does not vanish on $I_\delta \times \omega_\delta$ and therefore satisfies the equation $\mathcal{L}u = 0$ there. This is proved almost exactly as Theorem 2.2. The main difference is that here we do not want to shrink the ball B, so in order to prove the crucial estimate (2.9) we use instead (4.3) to show that the oscillation of φ on ω_δ is $\leq \delta/6$. In fact, for $|x' - x_0| \leq \delta$ and t, $t' \in \omega_\delta$, we have

$$|\varphi(x', t) - \varphi(x', t')| \leq |\varphi(x', t) - y_0| + |y_0 - \varphi(x', t')|.$$

Given $t \in \omega_\delta$, there exists $t_\bullet \in \mathcal{C}_q$ such that $|t - t_\bullet| < \delta$. Then

$$|\varphi(x',t) - y_0| = |\varphi(x',t) - \varphi(x_0,t_\bullet)| \le \frac{|x'-x_0| + |t-t_\bullet|}{24} \le \frac{\delta}{12}.$$

Similarly, $|y_0 - \varphi(x',t')| \le \delta/12$, so

$$|\varphi(x',t) - \varphi(x',t')| \le \frac{\delta}{6}, \quad |x'-x_0| \le \delta, \quad t,t' \in \omega_\delta.$$

Thus, we obtain

$$\frac{|x-x'|^2}{2} - |\varphi(x',t) - \varphi(x',t')|^2 \ge \frac{\delta^2}{16} - \frac{\delta^2}{36} \ge c > 0$$

for $|x - x_0| \le \delta/4$ and $|x' - x_0| \ge \delta/2$. The arguments in the proof of Theorem 2.2 allow us to show that $E_\tau u \to u$ uniformly on $I_\delta \times \omega_\delta$. As a corollary, we find a sequence of polynomials $P_j(z)$ that converge uniformly to U on $Z(I_\delta \times \omega_\delta)$ which is a neighborhood of z_0 because $t \mapsto \varphi(x_0,t)$ maps ω_δ onto an open interval that contains y_0. Therefore we conclude that $U(z)$ is holomorphic on a neighborhood of z_0.

Summing up, we have proved that the continuous function $U(z)$ is holomorphic on

$$D_1 = \{x + iy : \quad x \in I_1,\ m(x) < y < M(x)\}$$

except at the points where U vanishes. By the classical theorem of Radó, U is holomorphic everywhere in D_1, in particular it is holomorphic in a neighborhood of $z = 0$ and, since $u = U \circ Z$, this implies that the equation $\mathcal{L}u = 0$ is satisfied in a neighborhood of $(x,t) = (0,0)$.

Case 3

This is the general case and we make no restrictive assumption about the central point $p = (0,0)$, in particular, any of the inequalities $m(0) \le \varphi(0,0) \le M(0)$ may become an equality. It follows from the arguments in Case 2 that U is holomorphic on

$$D_j = \{x + iy : \quad x \in I_j,\ m(x) < y < M(x)\}, \quad 1 \le j \le N.$$

This already shows that $\mathcal{L}u = 0$ on $Z^{-1}(D_j)$ which, in general, is a proper subset of $I_j \times B$. To see that u is actually a homogeneous solution throughout $I_j \times B$ we apply Mergelyan's theorem: for fixed j, there exists a sequence of polynomials $P_k(z)$ that converges uniformly to U on $\overline{D}_j$. Thus, $\mathcal{L}(P_j \circ Z) = 0$ and $P_j(Z(x,t)) \to u(x,t)$ uniformly on $I_j \times B$, so $\mathcal{L}u = 0$ on $I_j \times B$ for every $1 \le j \le N$. Thus, we conclude that $\mathcal{L}u = 0$ on $\big((-a,a) \setminus \mathcal{N}\big) \times B$. The vector fields (2.4) may be written as

$$L_j = \frac{\partial}{\partial t_j} + \lambda_j \frac{\partial}{\partial x}, \quad \lambda_j = -\frac{i\varphi_{t_j}}{1 + i\varphi_x}, \quad j = 1, \ldots, n.$$

To complete the proof, we wish to show that

$$\int_{I\times B} u(x,t)L_j^t\psi(x,t)\,dxdt = 0$$

for any $\psi(x,t) \in C_0^\infty(I \times B)$. For each $0 < \epsilon < 1$, choose $\psi_\epsilon(x) \in C^\infty(-a,a)$ such that

1. $0 \le \psi_\epsilon(x) \le 1$;
2. $\psi_\epsilon(x) \equiv 1$ when $\operatorname{dist}(x,\mathcal{N}) \ge 2\epsilon$ and $\psi_\epsilon(x) \equiv 0$ for $\operatorname{dist}(x,\mathcal{N}) \le \epsilon$;
3. for some $C > 0$ independent of $0 < \epsilon < 1$, $|\psi'_\epsilon(x)| \le C\epsilon^{-1}$.

Since u is a solution on $\big(I \setminus \mathcal{N}\big) \times B$,

$$\begin{aligned} 0 &= \int_{I\times B} u(x,t)L_j^t(\psi_\epsilon(x)\psi(x,t))\,dxdt \\ &= \int_{I\times B} u(x,t)\psi_\epsilon(x)L_j^t(\psi)(x,t)\,dxdt \\ &\quad - \int_{I\times B} u(x,t)\lambda_j(x,t)\psi(x,t)\psi'_\epsilon(x)\,dxdt. \end{aligned}$$

Observe that since $\lambda_j(x,t) \equiv 0$ for $(x,t) \in \mathcal{N} \times B$ and $\psi'_\epsilon(x)$ is supported in the set $\{x \in \mathbb{R} :\ \epsilon \le \operatorname{dist}(x,\mathcal{N}) \le 2\epsilon\}$,

$$\lim_{\epsilon\to 0}\int_{I\times B} h(x,t)\lambda_j(x,t)\psi(x,t)\psi'_\epsilon(x)\,dxdt = 0,$$

while

$$\lim_{\epsilon\to 0}\int_{I\times B} u(x,t)\psi_\epsilon(x)L^t\psi(x,t)\,dxdt = \int_{I\times B} u(x,t)L^t\psi(x,t)\,dxdt.$$

It follows that $\int_{I\times B} u(x,t)L^t\psi(x,t)\,dxdt = 0$ and hence $\mathcal{L}u = 0$ holds on $I\times B$. □

5. An application to uniqueness

The Radó property can be used to give uniqueness in the Cauchy problem for continuous solutions without requiring any regularity for the initial "surface". Let $\mathcal{L}$ be smooth locally integrable structure of co-rank one defined on an open subset Ω of $\mathbb{R}^{n+1}$, $n \ge 1$, and let $U \subset \Omega$ be open. We denote by ∂U the set boundary points of U relative to Ω, i.e., $p \in \partial U$ if and only if $p \in \Omega$ and for every $\varepsilon > 0$, the ball $B(p,\varepsilon)$ contains both points of U and points of $\Omega \setminus U$. The orbit of $\mathcal{L}$ through the point p is defined as the orbit of p in the sense of Sussmann [Su] with respect to the set of real vector fields $\{X_\alpha\}$, with $X_\alpha = \Re L_\alpha$, where $\{L_\alpha\}$ is the set of all local smooth sections of $\mathcal{L}$ (we refer to [T1], [T2] and [BCH, Ch 3] for more information on orbits of locally integrable structures).

Definition 5.1. We say that ∂U is weakly noncharacteristic with respect to $\mathcal{L}$ if for every point $p \in \partial U$ the orbit of $\mathcal{L}$ through p intersects $\Omega \setminus \overline{U}$.

Recall that if $\Sigma = \partial U$ is a C^1 hypersurface, Σ is said to be noncharacteristic at $p \in \Sigma$ if $\Re L|_p$ is not tangent to Σ for some local smooth section of $\mathcal{L}$. This implies that the orbit of $\mathcal{L}$ through p must exit $\overline{U}$, so for regular surfaces the notion of "noncharacteristic" at every point is stronger than that of "weakly noncharacteristic". Similarly, if Σ is an orbit of $\mathcal{L}$ of dimension n that bounds some open set U, it will be a smooth hypersurface that fails to be noncharacteristic at every point and also fails to be weakly noncharacteristic. On the other hand, it is easy to give examples of a regular hypersurface Σ that bounds U and is characteristic precisely at one point p while the orbit through p eventually exits $\overline{U}$. In this case Σ will be weakly noncharacteristic although it fails to be noncharacteristic at every point.

Theorem 5.2. *Let $\mathcal{L}$ be a smooth locally integrable structure of co-rank one defined on an open subset Ω of $\mathbb{R}^{n+1}$, $n \geq 1$. Assume that $\mathcal{L}$ is locally solvable in degree one in Ω and that ∂U is weakly noncharacteristic. If u is continuous on $U \cup \partial U$, satisfies $\mathcal{L}u = 0$ on U in the weak sense and vanishes identically on ∂U, then there is an open set V, $\partial U \subset V \subset \Omega$ such that u vanishes identically on $V \cap U$.*

Proof. The proof is standard. Define $w \in C(\Omega)$ by extending u as zero on $\Omega \setminus (U \cup \partial U)$ (the continuity of w follows from the fact that u vanishes on ∂U). Then w is a Radó function and by Theorem 4.1 the equation $\mathcal{L}w = 0$ holds in Ω. By a classical application of the Baouendi-Treves approximation theorem the support of a homogeneous solution is $\mathcal{L}$-invariant, i.e., $S \doteq \operatorname{supp} w$ may be expressed as a union of orbits of $\mathcal{L}$ in Ω and the same holds for its complement, $\Omega \setminus S \supset \Omega \setminus \overline{U} \neq \emptyset$ (note that since ∂U is weakly noncharacteristic U is not dense in Ω). The fact that ∂U is weakly noncharacteristic implies that the union V of all the orbits of $\mathcal{L}$ that intersect $\Omega \setminus \overline{U}$ is an open set that contains ∂U on which w vanishes identically. □

Example. Consider in $\mathbb{R}^3$, where the coordinates are denoted by t_1, t_2, x, the function

$$Z(x,t) = x + i\,a(x)(t_1^2 + t_2^2)/2.$$

Here $a(x)$ is a smooth real function that is not real analytic at any point and vanishes exactly once at $x = 0$. Then $Z(x,t)$ is a global first integral of the system of vector fields

$$L_1 = \frac{\partial}{\partial t_1} - \frac{i\,t_1 a(x)}{1 + ia'(x)(t_1^2 + t_2^2)/2}\frac{\partial}{\partial x}$$
$$L_2 = \frac{\partial}{\partial t_2} - \frac{i\,t_2 a(x)}{1 + ia'(x)(t_1^2 + t_2^2)/2}\frac{\partial}{\partial x}$$

which span a structure $\mathcal{L}$ of co-rank one. This structure is locally solvable in degree one due to the fact that any nonempty fiber of Z over $\mathbb{R}^3$, $\mathcal{F}(x_0 + iy_0, \mathbb{R}^3)$, is either a circle contained in the hyperplane $x = x_0$ if $x_0 \neq 0$ or the whole hyperplane $x = 0$ if $x_0 = 0$, thus a connected set. If

$$U = \{(x, t_1, t_2) \in \mathbb{R}^3 : \quad t_1^3 > x\}$$

it follows that ∂U is weakly noncharacteristic with respect to $\mathcal{L}$ so theorem 5.2 can be applied in this situation. The choice of $a(x)$ also prevents the use of Holmgren's theorem even at noncharacteristic points.

Consider now a discrete set $D \subset U$ such that every point in ∂U is an accumulation point of D and set $U_1 = U \setminus D$. We have that $\partial U_1 = \partial U \cup D$ is not regular but it is still weakly noncharacteristic, so if a continuous function u satisfies

$$\begin{aligned} L_1 u &= 0 \quad \text{on } U_1 \\ L_2 u &= 0 \quad \text{on } U_1 \\ u &= 0 \quad \text{on } \partial U_1 \end{aligned}$$

then u must vanish identically on a neighborhood V of ∂U_1 and since $\mathbb{R}^3$ is the union of three orbits of $\mathcal{L}$, namely, $x > 0$, $x = 0$ and $x < 0$, it is apparent that u vanishes identically. Notice that L_1 and L_2 are Mizohata type vector fields and they are not locally solvable when considered individually, so uniqueness in the Cauchy problem for this example does not follow from the results in [HT].

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Applications of a Parametric Oka Principle for Liftings

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Dedicated to Linda P. Rothschild

Abstract. A parametric Oka principle for liftings, recently proved by Forstnerič, provides many examples of holomorphic maps that are fibrations in a model structure introduced in previous work of the author. We use this to show that the basic Oka property is equivalent to the parametric Oka property for a large class of manifolds. We introduce new versions of the basic and parametric Oka properties and show, for example, that a complex manifold X has the basic Oka property if and only if every holomorphic map to X from a contractible submanifold of $\mathbb{C}^n$ extends holomorphically to $\mathbb{C}^n$.

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1. Introduction

In this note, which is really an addendum to the author's papers [6] and [7], we use a new parametric Oka principle for liftings, very recently proved by Franc Forstnerič [4], to solve several problems left open in those papers. To make this note self-contained would require a large amount of technical background. Instead, we give a brief introduction and refer to the papers [2], [3], [6], [7] for more details.

The modern theory of the Oka principle began with Gromov's seminal paper [5] of 1989. Since then, researchers in Oka theory have studied more than a dozen so-called Oka properties that a complex manifold X may or may not have. These properties concern the task of deforming a continuous map f from a Stein manifold S to X into a holomorphic map. If this can always be done so that under the deformation f is kept fixed on a closed complex submanifold T of S on which f is holomorphic, then X is said to have the basic Oka property with interpolation

(BOPI). Equivalently (this is not obvious), T may be taken to be a closed analytic subvariety of a reduced Stein space S. If f can always be deformed to a holomorphic map so that the deformed maps stay arbitrarily close to f on a holomorphically convex compact subset K of S on which f is holomorphic and are holomorphic on a common neighbourhood of K, then X is said to have the basic Oka property with approximation (BOPA). If every holomorphic map to X from a compact convex subset K of $\mathbb{C}^n$ can be approximated uniformly on K by entire maps $\mathbb{C}^n \to X$, then X is said to have the convex approximation property (CAP), introduced in [3]. These properties all have parametric versions (POPI, POPA, PCAP) where instead of a single map f we have a family of maps depending continuously on a parameter.

Some of the basic and parametric Oka properties have been extended from complex manifolds to holomorphic maps (viewing a manifold as a constant map from itself). For example, a holomorphic map $f : X \to Y$ is said to satisfy POPI if for every Stein inclusion $T \hookrightarrow S$ (a Stein manifold S with a submanifold T), every finite polyhedron P with a subpolyhedron Q, and every continuous map $g : S \times P \to X$ such that the restriction $g|S \times Q$ is holomorphic along S, the restriction $g|T \times P$ is holomorphic along T, and the composition $f \circ g$ is holomorphic along S, there is a continuous map $G : S \times P \times I \to X$, where $I = [0, 1]$, such that:

1. $G(\cdot, \cdot, 0) = g$,
2. $G(\cdot, \cdot, 1) : S \times P \to X$ is holomorphic along S,
3. $G(\cdot, \cdot, t) = g$ on $S \times Q$ and on $T \times P$ for all $t \in I$,
4. $f \circ G(\cdot, \cdot, t) = f \circ g$ on $S \times P$ for all $t \in I$.

Equivalently, $Q \hookrightarrow P$ may be taken to be any cofibration between cofibrant topological spaces, such as the inclusion of a subcomplex in a CW-complex, and the existence of G can be replaced by the stronger statement that the inclusion into the space, with the compact-open topology, of continuous maps $h : S \times P \to X$ with $h = g$ on $S \times Q$ and on $T \times P$ and $f \circ h = f \circ g$ on $S \times P$ of the subspace of maps that are holomorphic along S is acyclic, that is, a weak homotopy equivalence (see [6], §16). (Here, the notion of cofibrancy for topological spaces and continuous maps is the stronger one that goes with Serre fibrations rather than Hurewicz fibrations. We remind the reader that a Serre fibration between smooth manifolds is a Hurewicz fibration, so we will simply call such a map a topological fibration.)

In [6], the category of complex manifolds was embedded into the category of prestacks on a certain simplicial site with a certain simplicial model structure such that all Stein inclusions are cofibrations, and a holomorphic map is acyclic if and only if it is topologically acyclic, and is a fibration if and only if it is a topological fibration and satisfies POPI. It was known then that complex manifolds with the geometric property of subellipticity satisfy POPI, but very few examples of nonconstant holomorphic maps satisfying POPI were known, leaving some doubt as to whether the model structure constructed in [6] is an appropriate homotopy-theoretic framework for the Oka principle. This doubt is dispelled by Forstnerič's parametric Oka principle for liftings.

Forstnerič has proved that the basic Oka properties for manifolds are equivalent, and that the parametric Oka properties for manifolds are equivalent ([2], see also [7], Theorem 1; this is not yet known for maps), so we can refer to them as the basic Oka property and the parametric Oka property, respectively. For Stein manifolds, the basic Oka property is equivalent to the parametric Oka property ([7], Theorem 2). In [7] (the comment following Theorem 5), it was noted that the equivalence of all the Oka properties could be extended to a much larger class of manifolds, including for example all quasi-projective manifolds, if we had enough examples of holomorphic maps satisfying POPI. This idea is carried out below. It remains an open problem whether the basic Oka property is equivalent to the parametric Oka property for all manifolds.

We conclude by introducing a new Oka property that we call the convex interpolation property, with a basic version equivalent to the basic Oka property and a parametric version equivalent to the parametric Oka property. In particular, we show that a complex manifold X has the basic Oka property if and only if every holomorphic map to X from a contractible submanifold of $\mathbb{C}^n$ extends holomorphically to $\mathbb{C}^n$. This is based on the proof of Theorem 1 in [7].

2. The parametric Oka principle for liftings

Using the above definition of POPI for holomorphic maps, we can state Forstnerič's parametric Oka principle for liftings, in somewhat less than its full strength, as follows.

Theorem 1 (Parametric Oka principle for liftings [4]**).** *Let X and Y be complex manifolds and $f : X \to Y$ be a holomorphic map which is either a subelliptic submersion or a holomorphic fibre bundle whose fibre has the parametric Oka property. Then f has the parametric Oka property with interpolation.*

The notion of a holomorphic submersion being subelliptic was introduced by Forstnerič [1], generalising the concept of ellipticity due to Gromov [5]. Subellipticity is the weakest currently-known sufficient geometric condition for a holomorphic map to satisfy POPI.

By a corollary of the main result of [6], Corollary 20, a holomorphic map f is a fibration in the so-called intermediate model structure constructed in [6] if and only if f is a topological fibration and satisfies POPI (and then f is a submersion). In particular, considering the case when f is constant, a complex manifold is fibrant if and only if it has the parametric Oka property. The following result is therefore immediate.

Theorem 2.

(1) *A subelliptic submersion is an intermediate fibration if and only if it is a topological fibration.*

(2) *A holomorphic fibre bundle is an intermediate fibration if and only if its fibre has the parametric Oka property.*

Part (1) is a positive solution to Conjecture 21 in [6]. For the only-if direction of (2), we simply take the pullback of the bundle by a map from a point into the base of the bundle and use the fact that in any model category, a pullback of a fibration is a fibration. As remarked in [6], (1) may be viewed as a new manifestation of the Oka principle, saying that for holomorphic maps satisfying the geometric condition of subellipticity, there is only a topological obstruction to being a fibration in the holomorphic sense defined by the model structure in [6]. Theorem 2 provides an ample supply of intermediate fibrations.

A result similar to our next theorem appears in [4]. The analogous result for the basic Oka property is Theorem 3 in [7].

Theorem 3. *Let X and Y be complex manifolds and $f : X \to Y$ be a holomorphic map which is an intermediate fibration.*

(1) *If Y satisfies the parametric Oka property, then so does X.*
(2) *If f is acyclic and X satisfies the parametric Oka property, then so does Y.*

Proof. (1) This follows immediately from the fact that if the target of a fibration in a model category is fibrant, so is the source.

(2) Let $T \hookrightarrow S$ be a Stein inclusion and $Q \hookrightarrow P$ an inclusion of parameter spaces. Let $h : S \times P \to Y$ be a continuous map such that the restriction $h|S \times Q$ is holomorphic along S and the restriction $h|T \times P$ is holomorphic along T. We need a lifting k of h by f with the same properties. Then the parametric Oka property of X allows us to deform k to a continuous map $S \times P \to X$ which is holomorphic along S, keeping the restrictions to $S \times Q$ and $T \times P$ fixed. Finally, we compose this deformation by f.

To obtain the lifting k, we first note that since f is an acyclic topological fibration, $h|T \times Q$ has a continuous lifting, which, since f satisfies POPI, may be deformed to a lifting which is holomorphic along T. We use the topological cofibration $T \times Q \hookrightarrow S \times Q$ to extend this lifting to a continuous lifting $S \times Q \to X$, which may be deformed to a lifting which is holomorphic along S, keeping the restriction to $T \times Q$ fixed. We do the same with $S \times Q$ replaced by $T \times P$ and get a continuous lifting of h restricted to $(S \times Q) \cup (T \times P)$ which is holomorphic along S on $S \times Q$ and along T on $T \times P$. Finally, we obtain k as a continuous extension of this lifting, using the topological cofibration $(S \times Q) \cup (T \times P) \hookrightarrow S \times P$. □

3. Equivalence of the basic and the parametric Oka properties

Following [7], by a *good* map we mean a holomorphic map which is an acyclic intermediate fibration, that is, a topological acyclic fibration satisfying POPI. We call a complex manifold X *good* if it is the target, and hence the image, of a good map from a Stein manifold. This map is then weakly universal in the sense that every holomorphic map from a Stein manifold to X factors through it.

A Stein manifold is obviously good. As noted in [7], the class of good manifolds is closed under taking submanifolds, products, covering spaces, finite branched covering spaces, and complements of analytic hypersurfaces. This does not take us beyond the class of Stein manifolds. However, complex projective space, and therefore every quasi-projective manifold, carries a holomorphic affine bundle whose total space is Stein (in algebraic geometry this observation is called the Jouanolou trick), and by Theorem 2, the bundle map is good. Therefore all quasi-projective manifolds are good. (A quasi-projective manifold is a complex manifold of the form $Y \setminus Z$, where Y is a projective variety and Z is a subvariety. We need the fact, proved using blow-ups, that Y can be taken to be smooth and Z to be a hypersurface.) The class of good manifolds thus appears to be quite large, but we do not know whether every manifold, or even every domain in $\mathbb{C}^n$, is good.

Theorem 4. *A good manifold has the basic Oka property if and only if it has the parametric Oka property.*

Proof. Let $S \to X$ be a good map from a Stein manifold S to a complex manifold X. If X has the basic Oka property, then so does S by [7], Theorem 3. Since S is Stein, S is elliptic by [7], Theorem 2, so S has the parametric Oka property. By Theorem 3, it follows that X has the parametric Oka property. $\square$

4. The convex interpolation property

Let us call a submanifold T of $\mathbb{C}^n$ *special* if T is the graph of a proper holomorphic embedding of a convex domain Ω in $\mathbb{C}^k$, $k \geq 1$, as a submanifold of $\mathbb{C}^{n-k}$, that is,

$$T = \{(x, \varphi(x)) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : x \in \Omega\},$$

where $\varphi : \Omega \to \mathbb{C}^{n-k}$ is a proper holomorphic embedding. We say that a complex manifold X satisfies the *convex interpolation property* (CIP) if every holomorphic map to X from a special submanifold T of $\mathbb{C}^n$ extends holomorphically to $\mathbb{C}^n$, that is, the restriction map $\mathscr{O}(\mathbb{C}^n, X) \to \mathscr{O}(T, X)$ is surjective.

We say that X satisfies the *parametric convex interpolation property* (PCIP) if whenever T is a special submanifold of $\mathbb{C}^n$, the restriction map $\mathscr{O}(\mathbb{C}^n, X) \to \mathscr{O}(T, X)$ is an acyclic Serre fibration. (Since $\mathbb{C}^n$ and T are holomorphically contractible, acyclicity is automatic; it is the fibration property that is at issue.) More explicitly, X satisfies PCIP if whenever T is a special submanifold of $\mathbb{C}^n$ and $Q \hookrightarrow P$ is an inclusion of parameter spaces, every continuous map $f : (\mathbb{C}^n \times Q) \cup (T \times P) \to X$, such that $f|\mathbb{C}^n \times Q$ is holomorphic along $\mathbb{C}^n$ and $f|T \times P \to X$ is holomorphic along T, extends to a continuous map $g : \mathbb{C}^n \times P \to X$ which is holomorphic along $\mathbb{C}^n$. The parameter space inclusions $Q \hookrightarrow P$ may range over all cofibrations of topological spaces or, equivalently, over the generating cofibrations $\partial B_n \hookrightarrow B_n$, $n \geq 0$, where B_n is the closed unit ball in $\mathbb{R}^n$ (we take B_0 to be a point and ∂B_0 to be empty). Clearly, CIP is PCIP with P a point and Q empty.

Lemma 1. *A complex manifold has the parametric convex interpolation property if and only if it has the parametric Oka property with interpolation for every inclusion of a special submanifold into $\mathbb{C}^n$.*

Proof. Using the topological acyclic cofibration $(\mathbb{C}^n \times Q) \cup (T \times P) \hookrightarrow \mathbb{C}^n \times P$, we can extend a continuous map $f : (\mathbb{C}^n \times Q) \cup (T \times P) \to X$ as in the definition of PCIP to a continuous map $g : \mathbb{C}^n \times P \to X$. POPI allows us to deform g to a continuous map $h : \mathbb{C}^n \times P \to X$ which is holomorphic along $\mathbb{C}^n$, keeping the restriction to $(\mathbb{C}^n \times Q) \cup (T \times P)$ fixed, so h extends f.

Conversely, if $h : \mathbb{C}^n \times P \to X$ is a continuous map such that $h|\mathbb{C}^n \times Q$ is holomorphic along $\mathbb{C}^n$ and $h|T \times P$ is holomorphic along T, and g is an extension of $f = h|(\mathbb{C}^n \times Q) \cup (T \times P)$ provided by PCIP, then the topological acyclic cofibration

$$\big(\big((\mathbb{C}^n \times Q) \cup (T \times P)\big) \times I\big) \cup \big(\mathbb{C}^n \times P \times \{0,1\}\big) \hookrightarrow \mathbb{C}^n \times P \times I$$

provides a deformation of h to g which is constant on $(\mathbb{C}^n \times Q) \cup (T \times P)$. □

Theorem 5. *A complex manifold has the convex interpolation property if and only if it has the basic Oka property. A complex manifold has the parametric interpolation property if and only if it has the parametric Oka property.*

Proof. We prove the equivalence of the parametric properties. The equivalence of the basic properties can be obtained by restricting the argument to the case when P is a point and Q is empty. By Lemma 1, POPI, which is one of the equivalent forms of the parametric Oka property by [2], Theorem 6.1, implies PCIP.

By [7], Theorem 1, POPI implies POPA (not only for manifolds but also for maps). The old version of POPA used in [7] does not require the intermediate maps to be holomorphic on a neighbourhood of the holomorphically convex compact subset in question, only arbitrarily close to the initial map. This property is easily seen to be equivalent to the current, ostensibly stronger version of POPA: see the comment preceding Lemma 5.1 in [2].

The proof of Theorem 1 in [7] shows that to prove POPA for K convex in $S = \mathbb{C}^k$, that is, to prove PCAP, it suffices to have POPI for Stein inclusions $T \hookrightarrow \mathbb{C}^n$ associated to convex domains Ω in $\mathbb{C}^k$ as in the definition of a special submanifold. Thus, by Lemma 1, PCIP implies PCAP, which is one of the equivalent forms of the parametric Oka property, again by [2], Theorem 6.1. □

There are many alternative definitions of a submanifold of $\mathbb{C}^n$ being special for which Theorem 5 still holds. For example, we could define special to mean topologically contractible: this is the weakest definition that obviously works. We could also define a submanifold of $\mathbb{C}^n$ to be special if it is biholomorphic to a bounded convex domain in $\mathbb{C}^k$, $k < n$. On the other hand, for the proof of Theorem 5 to go through, the class of special manifolds must contain T associated as above to every element Ω in some basis of convex open neighbourhoods of every convex compact subset K of $\mathbb{C}^n$ for every $n \geq 1$, such that K is of the kind termed *special* by Forstnerič (see [3], Section 1).

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Stability of the Vanishing of the $\overline{\partial}_b$-cohomology Under Small Horizontal Perturbations of the CR Structure in Compact Abstract q-concave CR Manifolds

Christine Laurent-Thiébaut

To Linda for her sixtieth birthday

Abstract. We consider perturbations of CR structures which preserve the complex tangent bundle. For a compact generic CR manifold its concavity properties and hence the finiteness of some $\overline{\partial}_b$-cohomology groups are also preserved by such perturbations of the CR structure. Here we study the stability of the vanishing of these groups.

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The tangential Cauchy-Riemann equation is one of the main tools in CR analysis and its properties are deeply related to the geometry of CR manifolds, in particular the complex tangential directions are playing an important role. For example it was noticed by Folland and Stein [3], when they studied the tangential Cauchy-Riemann operator on the Heisenberg group and more generally on strictly pseudoconvex real hypersurfaces of $\mathbb{C}^n$, that one get better estimates in the complex tangential directions. Therefore in the study of the stability properties for the tangential Cauchy-Riemann equation under perturbations of the CR structure it seems natural to consider perturbations which preserve the complex tangent vector bundle. Such perturbations can be represented as graphs in the complex tangent vector bundle over the original CR structure, they are defined by $(0,1)$-forms with values in the holomorphic tangent bundle. We call them horizontal perturbations.

We consider compact abstract CR manifolds and integrable perturbations of their CR structure which preserve their complex tangent vector bundle. Since the Levi form of a CR manifold depends only on its complex tangent vector bundle, such perturbations will preserve the Levi form and hence the concavity properties of the manifold which are closely related to the $\overline{\partial}_b$-cohomology. For example it is well known that, for q-concave compact CR manifolds of real dimension $2n-k$ and CR-dimension $n-k$, the $\overline{\partial}_b$-cohomology groups are finite dimensional in bidegree (p,r), when $1 \leq p \leq n$ and $1 \leq r \leq q-1$ or $n-k-q+1 \leq r \leq n-k$. Therefore, for a q-concave compact CR manifold, this finiteness property is stable by horizontal perturbations of the CR structure.

In this paper we are interested in the stability of the vanishing of the $\overline{\partial}_b$-cohomology groups after horizontal perturbations of the CR structure.

Let $\mathbb{M} = (\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class $\mathcal{C}^\infty$, of real dimension $2n-k$ and CR dimension $n-k$, and $\widehat{\mathbb{M}} = (\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ another abstract compact CR manifold such that $\widehat{H}_{0,1}\mathbb{M}$ is a smooth integrable horizontal perturbation of $H_{0,1}\mathbb{M}$, then $\widehat{H}_{0,1}\mathbb{M}$ is defined by a smooth form $\Phi \in \mathcal{C}^\infty_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$. We denote by $\overline{\partial}_b$ the tangential Cauchy-Riemann operator associated to the CR structure $H_{0,1}\mathbb{M}$ and by $\overline{\partial}_b^\Phi$ the tangential Cauchy-Riemann operator associated to the CR structure $\widehat{H}_{0,1}\mathbb{M}$.

The smooth $\overline{\partial}_b$-cohomology groups on $\mathbb{M}$ in bidegree $(0,r)$ and (n,r) are defined for $1 \leq r \leq n-k$ by:

$$H^{0,r}(\mathbb{M}) = \{f \in \mathcal{C}^\infty_{0,r}(\mathbb{M}) \mid \overline{\partial}_b f = 0\}/\overline{\partial}_b(\mathcal{C}^\infty_{0,r-1}(\mathbb{M}))$$

and

$$H^{n,r}(\mathbb{M}) = \{f \in \mathcal{C}^\infty_{n,r}(\mathbb{M}) \mid \overline{\partial}_b f = 0\}/\overline{\partial}_b(\mathcal{C}^\infty_{n,r-1}(\mathbb{M})).$$

If f is a smooth differential form of degree r, $1 \leq r \leq n-k$, on $\mathbb{M}$, we denote by $f_{r,0}$ its projection on the space $\mathcal{C}^\infty_{r,0}(\mathbb{M})$ of $(r,0)$-forms for the CR structure $H_{0,1}\mathbb{M}$. Note that if $r \geq n+1$ then $f_{r,0} = 0$.

The smooth $\overline{\partial}_b^\Phi$-cohomology groups on $\widehat{\mathbb{M}}$ in bidegree $(0,r)$ and (n,r) are defined for $1 \leq r \leq n-k$ by:

$$H^{0,r}_\Phi(\widehat{\mathbb{M}}) = \{f \in \mathcal{C}^\infty_{0,r}(\widehat{\mathbb{M}}) \mid f_{r,0} = 0,\ \overline{\partial}_b^\Phi f = 0\}/\overline{\partial}_b^\Phi(\mathcal{C}^\infty_{0,r-1}(\widehat{\mathbb{M}}))$$

and

$$H^{n,r}_\Phi(\widehat{\mathbb{M}}) = \{f \in \mathcal{C}^\infty_{n,r}(\widehat{\mathbb{M}}) \mid \overline{\partial}_b^\Phi f = 0\}/\overline{\partial}_b^\Phi(\mathcal{C}^\infty_{n,r-1}(\widehat{\mathbb{M}})).$$

In this paper the following stability result is proved:

Theorem 0.1. *Assume $\mathbb{M}$ is q-concave, there exists then a sequence $(\delta_l)_{l\in\mathbb{N}}$ of positive real numbers such that, if $\|\Phi\|_l < \delta_l$ for each $l \in \mathbb{N}$,*

- (i) *$H^{p,r-p}(\mathbb{M}) = 0$ for all $1 \leq p \leq r$ implies $H^{0,r}_\Phi(\widehat{\mathbb{M}}) = 0$ when $1 \leq r \leq q-2$, in the abstract case, and when $1 \leq r \leq q-1$, if $\mathbb{M}$ is locally embeddable,*
- (ii) *$H^{n-p,r+p}(\mathbb{M}) = 0$ for all $0 \leq p \leq n-k-r$ implies $H^{n,r}_\Phi(\widehat{\mathbb{M}}) = 0$ when $n-k-q+1 \leq r \leq n-k$.*

We also prove the stability of the solvability of the tangential Cauchy-Riemann equation with sharp anisotropic regularity (cf. Theorem 3.3).

Note that, when both CR manifolds $\mathbb{M}$ and $\widehat{\mathbb{M}}$ are embeddable in the same complex manifold (i.e., in the embedded case), a $(0,r)$-form f for the new CR structure $\widehat{H}_{0,1}\mathbb{M}$ is also a $(0,r)$-form for the original CR structure $H_{0,1}\mathbb{M}$ and hence the condition $f_{r,0} = 0$ in the definition of the cohomology groups $H^{0,r}_\Phi(\widehat{\mathbb{M}})$ is automatically fulfilled. In that case, Polyakov proved in [7] global homotopy formulas for a family of CR manifolds in small degrees which immediately imply the stability of the vanishing of the $\overline{\partial}_b$-cohomology groups of small degrees. In his paper he does not need the perturbation to preserve the complex tangent vector bundle, but his estimates are far to be sharp.

Moreover Polyakov [8] proved also that if a generically embedded compact CR manifold $M \subset X$ is at least 3-concave and satisfies $H^{0,1}(M, T'X_{|_M}) = 0$, then small perturbations of the CR structure are still embeddable in the same manifold X. From this result and the global homotopy formula from [7] one can derive some stability results for the vanishing of $\overline{\partial}_b$-cohomology groups of small degrees without hypothesis on the embeddability a priori of the perturbed CR structure.

The main interest of our paper is that we do not assume the CR manifolds $\mathbb{M}$ and $\widehat{\mathbb{M}}$ to be embeddable. In the case where the original CR manifold $\mathbb{M}$ is embeddable it covers the case where it is unknown if the perturbed CR structure is embeddable in the same manifold as the original one, for example when the manifold $\mathbb{M}$ is only 2-concave. Finally we also reach the case of the $\overline{\partial}_b$-cohomology groups of large degrees, which, even in the embedded case, cannot be deduced from the works of Polyakov.

The main tool in the proof of the stability of the vanishing of the $\overline{\partial}_b$-cohomology groups is a fixed point theorem which is derived from global homotopy formulas with sharp anisotropic estimates. Such formulas are proved in [9], in the abstract case, by using the L^2 theory for the $\Box_b$ operator and in [5], in the locally embeddable case, by first proving that the integral operators associated to the kernels built in [1] satisfy sharp anisotropic estimates, which implies local homotopy formulas with sharp anisotropic estimates, and then by using the globalization method from [6] and [2].

1. CR structures

Let $\mathbb{M}$ be a $\mathcal{C}^l$-smooth, $l \geq 1$, paracompact differential manifold, we denote by $T\mathbb{M}$ the tangent bundle of $\mathbb{M}$ and by $T_\mathbb{C}\mathbb{M} = \mathbb{C} \otimes T\mathbb{M}$ the complexified tangent bundle.

Definition 1.1. An *almost CR structure* on $\mathbb{M}$ is a subbundle $H_{0,1}\mathbb{M}$ of $T_\mathbb{C}\mathbb{M}$ such that $H_{0,1}\mathbb{M} \cap \overline{H_{0,1}\mathbb{M}} = \{0\}$.

If the almost CR structure is integrable, i.e., for all $Z, W \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$ then $[Z, W] \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$, then it is called a *CR structure*.

If $H_{0,1}\mathbb{M}$ is a CR structure, the pair $(\mathbb{M}, H_{0,1}\mathbb{M})$ is called an *abstract CR manifold*.

The *CR dimension* of $\mathbb{M}$ is defined by CR-dim $\mathbb{M} = \mathrm{rk}_\mathbb{C}\ \mathrm{H}_{0,1}\mathbb{M}$.

We set $H_{1,0}\mathbb{M} = \overline{H_{0,1}\mathbb{M}}$ and we denote by $H^{0,1}\mathbb{M}$ the dual bundle $(H_{0,1}\mathbb{M})^*$ of $H_{0,1}\mathbb{M}$.

Let $\Lambda^{0,q}\mathbb{M} = \bigwedge^q(H^{0,1}\mathbb{M})$, then $\mathcal{C}^s_{0,q}(\mathbb{M}) = \Gamma^s(\mathbb{M}, \Lambda^{0,q}\mathbb{M})$ is called the space of $(0,q)$-forms of class $\mathcal{C}^s$, $0 \leq s \leq l$, on $\mathbb{M}$.

We define $\Lambda^{p,0}\mathbb{M}$ as the space of forms of degree p that annihilate any p-vector on $\mathbb{M}$ that has more than one factor contained in $H_{0,1}\mathbb{M}$. Then $\mathcal{C}^s_{p,q}(\mathbb{M}) = \mathcal{C}^s_{0,q}(\mathbb{M}, \Lambda^{p,0}\mathbb{M})$ is the space of $(0,q)$-forms of class $\mathcal{C}^s$ with values in $\Lambda^{p,0}\mathbb{M}$.

If the almost CR structure is a CR structure, i.e., if it is integrable, and if $s \geq 1$, then we can define an operator

$$\overline{\partial}_b \ : \ \mathcal{C}^s_{0,q}(\mathbb{M}) \to \mathcal{C}^{s-1}_{0,q+1}(\mathbb{M}), \tag{1.1}$$

called the *tangential Cauchy-Riemann operator*, by setting $\overline{\partial}_b f = df_{|_{H_{0,1}\mathbb{M}\times\cdots\times H_{0,1}\mathbb{M}}}$. It satisfies $\overline{\partial}_b \circ \overline{\partial}_b = 0$.

Definition 1.2. Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract CR manifold, X be a complex manifold and $F \ : \ \mathbb{M} \to X$ be an embedding of class $\mathcal{C}^l$, then F is called a *CR embedding* if $dF(H_{0,1}\mathbb{M})$ is a subbundle of the bundle $T_{0,1}X$ of the antiholomorphic vector fields of X and $dF(H_{0,1}\mathbb{M}) = T_{0,1}X \cap T_{\mathbb{C}}F(\mathbb{M})$.

Let F be a CR embedding of an abstract CR manifold $\mathbb{M}$ into a complex manifold X and set $M = F(\mathbb{M})$, then M is a CR manifold with the CR structure $H_{0,1}M = T_{0,1}X \cap T_{\mathbb{C}}M$.

Let U be a coordinate domain in X, then $F_{|_{F^{-1}(U)}} = (f_1, \ldots, f_N)$, with $N = \dim_{\mathbb{C}}\mathrm{X}$, and F is a CR embedding if and only if, for all $1 \leq j \leq N$, $\overline{\partial}_b f_j = 0$.

A CR embedding is called *generic* if $\dim_{\mathbb{C}} \mathrm{X} - \mathrm{rk}_{\mathbb{C}}\ \mathrm{H}_{0,1}\mathrm{M} = \mathrm{codim}_{\mathbb{R}}\ \mathrm{M}$.

Definition 1.3. An almost CR structure $\widehat{H}_{0,1}\mathbb{M}$ on $\mathbb{M}$ is said to be of *finite distance* to a given CR structure $H_{0,1}\mathbb{M}$ if $\widehat{H}_{0,1}\mathbb{M}$ can be represented as a graph in $T_{\mathbb{C}}\mathbb{M}$ over $H_{0,1}\mathbb{M}$.

It is called an *horizontal perturbation* of the CR structure $H_{0,1}\mathbb{M}$ if it is of finite distance to $H_{0,1}\mathbb{M}$ and moreover there exists $\Phi \in \mathcal{C}_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$ such that

$$\widehat{H}_{0,1}\mathbb{M} = \{\overline{W} \in T_{\mathbb{C}}\mathbb{M} \mid \overline{W} = \overline{Z} - \Phi(\overline{Z}), \overline{Z} \in H_{0,1}\mathbb{M}\}, \tag{1.2}$$

which means that $\widehat{H}_{0,1}\mathbb{M}$ is a graph in $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ over $H_{0,1}\mathbb{M}$.

Note that an horizontal perturbation of the original CR structure preserves the complex tangent bundle $H\mathbb{M}$.

Assume $\mathbb{M}$ is an abstract CR manifold and $\widehat{H}_{0,1}\mathbb{M}$ is an integrable horizontal perturbation of the original CR structure $H_{0,1}\mathbb{M}$ on $\mathbb{M}$. If $\overline{\partial}_b^{\Phi}$ denotes the tangential Cauchy-Riemann operator associated to the CR structure $\widehat{H}_{0,1}\mathbb{M}$, then we have

$$\overline{\partial}_b^{\Phi} = \overline{\partial}_b - \Phi \lrcorner d = \overline{\partial}_b - \Phi \lrcorner \partial_b, \tag{1.3}$$

where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator associated to the original CR structure $H_{0,1}\mathbb{M}$ and ∂_b involves only holomorphic tangent vector fields.

The annihilator $H^0\mathbb{M}$ of $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ in $T^*_{\mathbb{C}}\mathbb{M}$ is called the *characteristic bundle* of $\mathbb{M}$. Given $p \in \mathbb{M}$, $\omega \in H^0_p\mathbb{M}$ and $X, Y \in H_p\mathbb{M}$, we choose $\widetilde{\omega} \in \Gamma(\mathbb{M}, H^0\mathbb{M})$ and $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathbb{M}, H\mathbb{M})$ with $\widetilde{\omega}_p = \omega$, $\widetilde{X}_p = X$ and $\widetilde{Y}_p = Y$. Then $d\widetilde{\omega}(X, Y) = -\omega([\widetilde{X}, \widetilde{Y}])$. Therefore we can associate to each $\omega \in H^0_p\mathbb{M}$ an hermitian form

$$L_\omega(X) = -i\omega([\widetilde{X}, \overline{\widetilde{X}}]) \tag{1.4}$$

on $H_p\mathbb{M}$. This is called the *Levi form* of $\mathbb{M}$ at $\omega \in H^0_p\mathbb{M}$.

In the study of the $\overline{\partial}_b$-complex two important geometric conditions were introduced for CR manifolds of real dimension $2n - k$ and CR-dimension $n - k$. The first one by Kohn in the hypersurface case, $k = 1$, the condition Y(q), the second one by Henkin in codimension k, $k \geq 1$, the q-concavity.

An abstract CR manifold $\mathbb{M}$ of hypersurface type satisfies Kohn's condition $Y(q)$ at a point $p \in \mathbb{M}$ for some $0 \leq q \leq n-1$, if the Levi form of $\mathbb{M}$ at p has at least $\max(n-q, q+1)$ eigenvalues of the same sign or at least $\min(n-q, q+1)$ eigenvalues of opposite signs.

An abstract CR manifold $\mathbb{M}$ is said to be *q-concave* at $p \in \mathbb{M}$ for some $0 \leq q \leq n-k$, if the Levi form L_ω at $\omega \in H^0_p\mathbb{M}$ has at least q negative eigenvalues on $H_p\mathbb{M}$ for every nonzero $\omega \in H^0_p\mathbb{M}$.

In [9] the condition Y(q) is extended to arbitrary codimension.

Definition 1.4. An abstract CR manifold is said to satisfy *condition Y(q)* for some $1 \leq q \leq n-k$ at $p \in \mathbb{M}$ if the Levi form L_ω at $\omega \in H^0_p\mathbb{M}$ has at least $n-k-q+1$ positive eigenvalues or at least $q+1$ negative eigenvalues on $H_p\mathbb{M}$ for every nonzero $\omega \in H^0_p\mathbb{M}$.

Note that in the hypersurface case, i.e., $k = 1$, this condition is equivalent to the classical condition Y(q) of Kohn for hypersurfaces. Moreover, if $\mathbb{M}$ is q-concave at $p \in \mathbb{M}$, then $q \leq (n-k)/2$ and condition Y(r) is satisfied at $p \in \mathbb{M}$ for any $0 \leq r \leq q-1$ and $n-k-q+1 \leq r \leq n-k$.

2. Stability of vanishing theorems by horizontal perturbations of the CR structure

Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class $\mathcal{C}^\infty$, of real dimension $2n-k$ and CR dimension $n-k$, and $\widehat{H}_{0,1}\mathbb{M}$ be an integrable horizontal perturbation of $H_{0,1}\mathbb{M}$. We denote by $\mathbb{M}$ the abstract CR manifold $(\mathbb{M}, H_{0,1}\mathbb{M})$ and by $\widehat{\mathbb{M}}$ the abstract CR manifold $(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$.

Since $\widehat{H}_{0,1}\mathbb{M}$ is an horizontal perturbation of $H_{0,1}\mathbb{M}$, which means that $\widehat{H}_{0,1}\mathbb{M}$ is a graph in $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ over $H_{0,1}\mathbb{M}$, the space $H\widehat{\mathbb{M}} = H_{1,0}\widehat{\mathbb{M}} \oplus H_{0,1}\widehat{\mathbb{M}}$ coincides with the space $H\mathbb{M}$ and consequently the two abstract CR manifolds $\mathbb{M}$ and $\widehat{\mathbb{M}}$ have the same characteristic bundle and hence the same Levi form. This implies in particular that if $\mathbb{M}$ satisfies condition Y(q) at each point, then $\widehat{\mathbb{M}}$ satisfies also condition Y(q) at each point.

It follows from the Hodge decomposition theorem and the results in [9] that if $\mathbb{M}$ is an abstract compact CR manifold of class $\mathcal{C}^\infty$ which satisfies condition Y(q) at each point, then the cohomology groups $H^{p,q}(\mathbb{M})$, $0 \leq p \leq n$, are finite dimensional. A natural question is then the stability by small horizontal perturbations of the CR structure of the vanishing of these groups.

Let us consider a sequence $(\mathcal{B}^l(\mathbb{M}),\ l \in \mathbb{N})$ of Banach spaces with $\mathcal{B}^{l+1}(\mathbb{M}) \subset \mathcal{B}^l(\mathbb{M})$, which are invariant by horizontal perturbations of the CR structure of $\mathbb{M}$ and such that if $f \in \mathcal{B}^l(\mathbb{M})$, $l \geq 1$, then $X_{\mathbb{C}} f \in \mathcal{B}^{l-1}(\mathbb{M})$ for all complex tangent vector fields $X_{\mathbb{C}}$ to $\mathbb{M}$ and there exists $\theta(l) \in \mathbb{N}$ with $\theta(l+1) \geq \theta(l)$ such that $fg \in \mathcal{B}^l(\mathbb{M})$ if $f \in \mathcal{B}^l(\mathbb{M})$ and $g \in \mathcal{C}^{\theta(l)}(\mathbb{M})$. Such a sequence $(\mathcal{B}^l(\mathbb{M}),\ l \in \mathbb{N})$ will be called *a sequence of anisotropic spaces.* We denote by $\mathcal{B}^l_{p,r}(\mathbb{M})$ the space of (p,r)-forms on $\mathbb{M}$ whose coefficients belong to $\mathcal{B}^l(\mathbb{M})$. Moreover we will say that these Banach spaces are *adapted to the $\overline{\partial}_b$-equation in degree $r \geq 1$* if, when $H^{p,r}(\mathbb{M}) = 0$, $0 \leq p \leq n$, there exist linear continuous operators A_s, $s = r, r+1$, from $\mathcal{B}^0_{p,s}(\mathbb{M})$ into $\mathcal{B}^0_{p,s-1}(\mathbb{M})$ which are also continuous from $\mathcal{B}^l_{p,s}(\mathbb{M})$ into $\mathcal{B}^{l+1}_{p,s-1}(\mathbb{M})$, $l \in \mathbb{N}$, and moreover satisfy

$$f = \overline{\partial}_b A_r f + A_{r+1} \overline{\partial}_b f\,, \tag{2.1}$$

for $f \in \mathcal{B}^l_{p,r}(\mathbb{M})$.

Theorem 2.1. *Let $\mathbb{M} = (\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class $\mathcal{C}^\infty$, of real dimension $2n-k$ and CR dimension $n-k$, and $\widehat{\mathbb{M}} = (\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ another abstract compact CR manifold such that $\widehat{H}_{0,1}\mathbb{M}$ is an integrable horizontal perturbation of $H_{0,1}\mathbb{M}$. Let also $(\mathcal{B}^l(\mathbb{M}),\ l \in \mathbb{N})$ be a sequence of anisotropic Banach spaces and q be an integer, $1 \leq q \leq (n-k)/2$. Finally let $\Phi \in \mathcal{C}^{\theta(l)}_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$ be the differential form which defines the tangential Cauchy-Riemann operator $\overline{\partial}^\Phi_b = \overline{\partial}_b - \Phi \lrcorner \partial_b$ associated to the CR structure $\widehat{H}_{0,1}\mathbb{M}$.*

Assume $H^{p,r-p}(\mathbb{M}) = 0$, for $1 \leq p \leq r$ and $1 \leq r \leq s_1(q)$, or $H^{n-p,r+p}(\mathbb{M}) = 0$, for $0 \leq p \leq n-k-r$ and $s_2(q) \leq r \leq n-k$ and that the Banach spaces $(\mathcal{B}^l(\mathbb{M}),\ l \in \mathbb{N})$ are adapted to the $\overline{\partial}_b$-equation in degree r, $1 \leq r \leq s_1(q)$ or $s_2(q) \leq r \leq n-k$. Then, for each $l \in \mathbb{N}$, there exists $\delta > 0$ such that, if $\|\Phi\|_{\theta(l)} < \delta$,

(i) *for each $\overline{\partial}^\phi_b$-closed form f in $\mathcal{B}^l_{0,r}(\widehat{\mathbb{M}})$, $1 \leq r \leq s_1(q)$, such that the part of f of bidegree $(0,r)$ for the initial CR structure $H_{0,1}\mathbb{M}$ vanishes, there exists a form u in $\mathcal{B}^{l+1}_{0,r-1}(\widehat{\mathbb{M}})$ satisfying $\overline{\partial}^\phi_b u = f$,*

(ii) *for each $\overline{\partial}^\phi_b$-closed form f in $\mathcal{B}^l_{n,r}(\widehat{\mathbb{M}})$, $s_2(q) \leq r \leq n-k$, there exists a form u in $\mathcal{B}^{l+1}_{n,r-1}(\widehat{\mathbb{M}})$ satisfying $\overline{\partial}^\phi_b u = f$.*

Remark 2.2. Note that if both $\mathbb{M} = (\mathbb{M}, H_{0,1}\mathbb{M})$ and $\widehat{\mathbb{M}} = (\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ are embeddable in the same complex manifold X, any r-form on the differential manifold $\mathbb{M}$, which represents a form of bidegree $(0,r)$ for the CR structure $\widehat{H}_{0,1}\mathbb{M}$ represents also a form of bidegree $(0,r)$ for the CR structure $H_{0,1}\mathbb{M}$. Hence the bidegree hypothesis in (i) of Theorem 2.1 is automatically fulfilled.

Proof. Let $f \in \mathcal{B}^l_{0,r}(\widehat{\mathbb{M}})$ be a $(0,r)$-form for the CR structure $\widehat{H}_{0,1}\mathbb{M}$, $1 \leq r \leq s_1(q)$, such that $\overline{\partial}_b^{\Phi} f = 0$, we want to solve the equation

$$\overline{\partial}_b^{\Phi} u = f. \tag{2.2}$$

The form f can be written $\sum_{p=0}^{r} f_{p,r-p}$, where the forms $f_{p,r-p}$ are of type $(p, r-p)$ for the CR structure $H_{0,1}\mathbb{M}$. Then by considerations of bidegrees, the equation $\overline{\partial}_b^{\Phi} f = 0$ is equivalent to the family of equations $\overline{\partial}_b^{\Phi} f_{p,r-p} = 0$, $0 \leq p \leq r$.

Moreover, if $u = \sum_{s=0}^{r-1} u_{s,r-1-s}$, where the forms $u_{s,r-1-s}$ are of type $(s, r-1-s)$ for the CR structure $H_{0,1}\mathbb{M}$, is a solution of (2.2), then

$$\overline{\partial}_b^{\Phi} u_{p,r-1-p} = f_{p,r-p}\,,$$

for $0 \leq p \leq r-1$, and

$$f_{r,0} = 0\,.$$

Therefore a necessary condition on f for the solvability of (2.2) is that $\overline{\partial}_b^{\Phi} f = 0$ and $f_{r,0} = 0$, where $f_{r,0}$ is the part of type $(r,0)$ of f for the CR structure $H_{0,1}\mathbb{M}$, and, to solve (2.2), we have to consider the equation

$$\overline{\partial}_b^{\Phi} v = g, \tag{2.3}$$

where $g \in \mathcal{B}^l_{p,r-p}(\mathbb{M})$ is a $(p, r-p)$-form for the CR structure $H_{0,1}\mathbb{M}$, $0 \leq p \leq r-1$, which is $\overline{\partial}_b^{\Phi}$-closed. By definition of the operator $\overline{\partial}_b^{\Phi}$, this means solving the equation $\overline{\partial}_b v = g + \Phi \lrcorner \partial_b v$. Consequently if v is a solution of (2.3), then $\overline{\partial}_b(g + \Phi \lrcorner \partial_b v) = 0$ and by (2.1)

$$\overline{\partial}_b(A_{r-p}(g + \Phi \lrcorner \partial_b v)) = g + \Phi \lrcorner \partial_b v.$$

Assume Φ is of class $\mathcal{C}^{\theta(l)}$, then the map

$$\begin{array}{rcll} \Theta \ : \ \mathcal{B}^{l+1}_{p,r-1}(\mathbb{M}) & \rightarrow & \mathcal{B}^{l+1}_{p,r-1}(\mathbb{M}) \\ v & \mapsto & A_{r-p} g + A_{r-p}(\Phi \lrcorner \partial_b v)\,. \end{array}$$

is continuous, and the fixed points of Θ are good candidates to be solutions of (2.3).

Let δ_0 such that, if $\|\Phi\|_{\theta(l)} < \delta_0$, then the norm of the bounded endomorphism $A_{r-p} \circ \Phi \lrcorner \partial_b$ of $\mathcal{B}^{l+1}_{p,r-p-1}(\mathbb{M})$ is equal to $\epsilon_0 < 1$. We shall prove that, if $\|\Phi\|_{\theta(l)} < \delta_0$, Θ admits a unique fixed point.

Consider first the uniqueness of the fixed point. Assume v_1 and v_2 are two fixed points of Θ, then

$$\begin{array}{rcl} v_1 & = & \Theta(v_1) = A_{r-p} g + A_{r-p}(\Phi \lrcorner \partial_b v_1) \\ v_2 & = & \Theta(v_2) = A_{r-p} g + A_{r-p}(\Phi \lrcorner \partial_b v_2). \end{array}$$

This implies

$$v_1 - v_2 = A_{r-p}\big(\Phi \lrcorner \partial_b (v_1 - v_2)\big)$$

and, by the hypothesis on Φ,

$$\|v_1 - v_2\|_{\mathcal{B}^{l+1}} < \|v_1 - v_2\|_{\mathcal{B}^{l+1}} \quad \text{or} \quad v_1 = v_2$$

and hence $v_1 = v_2$.

For the existence we proceed by iteration. We set $v_0 = \Theta(0) = A_{r-p}(g)$ and, for $n \geq 0$, $v_{n+1} = \Theta(v_n)$. Then for $n \geq 0$, we get

$$v_{n+1} - v_n = A_{r-p}(\Phi \lrcorner \partial_b (v_n - v_{n-1})).$$

Therefore, if $\|\Phi\|_{\theta(l)} < \delta_0$, the sequence $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{B}^{l+1}_{p,r-1}(\mathbb{M})$ and hence converges to a form v, moreover by continuity of the map Θ, v satisfies $\Theta(v) = v$.

It remains to prove that v is a solution of (2.3). Since $H^{p,r-p}(\mathbb{M}) = 0$ for $1 \leq p \leq r$, it follows from (2.1) and from the definition of the sequence $(v_n)_{n\in\mathbb{N}}$ that

$$g - \overline{\partial}_b^{\Phi} v_{n+1} = \Phi \lrcorner \partial_b (v_{n+1} - v_n) + A_{r-p+1} \overline{\partial}_b (g + \Phi \lrcorner \partial_b v_n)$$

and since

$$\begin{aligned}
\overline{\partial}_b (g + \Phi \lrcorner \partial_b v_n) &= \overline{\partial}_b g - \overline{\partial}_b (\overline{\partial}_b - \Phi \lrcorner \partial_b) v_n \\
&= \overline{\partial}_b g - \overline{\partial}_b (\overline{\partial}_b^{\Phi} v_n) \\
&= \overline{\partial}_b g - (\overline{\partial}_b^{\Phi} + \Phi \lrcorner \partial_b)(\overline{\partial}_b^{\Phi} v_n) \\
&= \overline{\partial}_b g - \Phi \lrcorner \partial_b (\overline{\partial}_b^{\Phi} v_n), \quad \text{since } (\overline{\partial}_b^{\Phi})^2 = 0 \\
&= \Phi \lrcorner \partial_b (g - \overline{\partial}_b^{\Phi} v_n), \quad \text{since } \overline{\partial}_b^{\Phi} g = 0,
\end{aligned}$$

we get

$$g - \overline{\partial}_b^{\Phi} v_{n+1} = \Phi \lrcorner \partial_b (v_{n+1} - v_n) + A_{r-p+1}(\Phi \lrcorner \partial_b (g - \overline{\partial}_b^{\Phi} v_n)). \tag{2.4}$$

Note that since $g \in \mathcal{B}^l_{p,r}(\mathbb{M})$ and Φ is of class $\mathcal{C}^{\theta(l)}$, it follows from the definition of the v_ns that $v_n \in \mathcal{B}^{l+1}_{p,r}(\mathbb{M})$ and $\overline{\partial}_b^{\Phi} v_n \in \mathcal{B}^l_{p,r}(\mathbb{M})$ for all $n \in \mathbb{N}$.

Thus by (2.4), we have the estimate

$$\|g - \overline{\partial}_b^{\Phi} v_{n+1}\|_{\mathcal{B}^l} \leq \|\Phi \lrcorner \partial_b\| \|(v_{n+1} - v_n)\|_{\mathcal{B}^{l+1}} + \|A_{r-p+1} \circ \Phi \lrcorner \partial_b\| \|g - \overline{\partial}_b^{\Phi} v_n\|_{\mathcal{B}^l}. \tag{2.5}$$

Let δ such that if $\|\Phi\|_{\theta(l)} < \delta$, then the maximum of the norm of the bounded endomorphisms $A_s \circ \Phi \lrcorner \partial_b$, $s = r-p, r-p+1$, of $\mathcal{B}^l_{p,s-1}(\mathbb{M})$ is equal to $\epsilon < 1$. Assume $\|\Phi\|_{\theta(l)} < \delta$, then by induction we get

$$\|g - \overline{\partial}_b^{\Phi} v_{n+1}\|_{\mathcal{B}^l} \leq (n+1)\epsilon^{n+1} \|\Phi \lrcorner \partial_b\| \|v_0\|_{\mathcal{B}^{l+1}} + \epsilon^{n+1} \|g - \overline{\partial}_b^{\Phi} v_0\|_{\mathcal{B}^l}. \tag{2.6}$$

But

$$g - \overline{\partial}_b^{\Phi} v_0 = \Phi \lrcorner \partial_b A_{r-p} g + A_{r-p+1}(\Phi \lrcorner \partial_b g)$$

and hence

$$\|g - \overline{\partial}_b^{\Phi} v_0\|_{\mathcal{B}^l} \leq \|\Phi \lrcorner \partial_b\| \|A_{r-p} g\|_{\mathcal{B}^{l+1}} + \epsilon \|g\|_{\mathcal{B}^l}.$$

This implies

$$\|g - \overline{\partial}_b^{\Phi} v_{n+1}\|_{\mathcal{B}^l} \leq (n+2)\epsilon^{n+1} \|\Phi \lrcorner \partial_b\| \|A_{r-p}\| \|g\|_{\mathcal{B}^l} + \epsilon^{n+2} \|g\|_{\mathcal{B}^l}. \tag{2.7}$$

Since $\epsilon < 1$, the right-hand side of (2.7) tends to zero, when n tends to infinity and by continuity of the operator $\overline{\partial}_b^{\Phi}$ from $\mathcal{B}^{l+1}_{p,r-1}(\mathbb{M})$ into $\mathcal{B}^l_{p,r}(\mathbb{M})$, the left-hand side of (2.7) tends to $\|g - \overline{\partial}_b^{\Phi} v\|_{\mathcal{B}^l}$, when n tends to infinity, which implies that v is a solution of (2.3).

Now if $f \in \mathcal{B}^l_{n,r}(\widehat{\mathbb{M}})$ is an (n, r)-form for the CR structure $\widehat{H}_{0,1}\mathbb{M}$, $s_2(q) \leq r \leq n-k$, such that $\overline{\partial}_b^{\Phi} f = 0$, then the form f can be written $\sum_{p=0}^{n-k-r} f_{n-p,r+p}$, where the forms $f_{n-p,r+p}$ are $\overline{\partial}_b^{\Phi}$-closed and of type $(n-p, r+p)$ for the CR structure $H_{0,1}\mathbb{M}$. Then to solve the equation

$$\overline{\partial}_b^{\Phi} u = f,$$

it is sufficient to solve the equation $\overline{\partial}_b^{\Phi} v = g$ for $g \in \mathcal{B}^l_{n-p,r+p}(\mathbb{M})$ and this can be done in the same way as in the case of the small degrees, but using the vanishing of the cohomology groups $H^{n-p,p+r}(\mathbb{M})$ for $0 \leq p \leq n-k-r$ and $s_2(q) \leq r \leq n-k$. □

Assume the horizontal perturbation of the original CR structure on $\mathbb{M}$ is smooth, i.e., Φ is of class $\mathcal{C}^\infty$, then we can defined on $\widehat{\mathbb{M}}$ the cohomology groups

$$H^{o,r}_{\Phi}(\widehat{\mathbb{M}}) = \{f \in \mathcal{C}^\infty_{o,r}(\widehat{\mathbb{M}}) \mid f_{r,0} = 0,\ \overline{\partial}_b^{\Phi} f = 0\} / \overline{\partial}_b^{\Phi}(\mathcal{C}^\infty_{o,r-1}(\widehat{\mathbb{M}}))$$

and

$$H^{n,r}_{\Phi}(\widehat{\mathbb{M}}) = \{f \in \mathcal{C}^\infty_{o,r}(\widehat{\mathbb{M}}) \mid \overline{\partial}_b^{\Phi} f = 0\} / \overline{\partial}_b^{\Phi}(\mathcal{C}^\infty_{n,r-1}(\widehat{\mathbb{M}}))$$

for $1 \leq r \leq n-k$.

Corollary 2.3. *Under the hypotheses of Theorem* 2.1*, if the sequence* $(\mathcal{B}^l(\mathbb{M}),\ l \in \mathbb{N})$ *of anisotropic Banach spaces is such that* $\cap_{l\in\mathbb{N}} B_l(\mathbb{M}) = \mathcal{C}^\infty(\mathbb{M})$ *and if the horizontal perturbation of the original CR structure on* $\mathbb{M}$ *is smooth, there exists a sequence* $(\delta_l)_{l\in\mathbb{N}}$ *of positive real numbers such that, if* $\|\Phi\|_{\theta(l)} < \delta_l$ *for each* $l \in \mathbb{N}$

(i) $H^{p,r-p}(\mathbb{M}) = 0$*, for all* $1 \leq p \leq r$*, implies* $H^{o,r}_{\Phi}(\widehat{\mathbb{M}}) = 0$*, when* $1 \leq r \leq s_1(q)$*,*

(ii) $H^{n-p,r+p}(\mathbb{M}) = 0$*, for all* $0 \leq p \leq n-k-r$*, implies* $H^{n,r}_{\Phi}(\widehat{\mathbb{M}}) = 0$*, when* $s_2(q) \leq r \leq n-k$*.*

Proof. It is a direct consequence of the proof of Theorem 2.1 by the uniqueness of the fixed point of Θ. □

3. Anisotropic spaces

In the previous section the main theorem is proved under the assumption of the existence of sequences of anisotropic spaces on abstract CR manifolds satisfying good properties with respect to the tangential Cauchy-Riemann operator. We will make precise Theorem 2.1 by considering some Sobolev and some Hölder anisotropic spaces for which global homotopy formulas for the tangential Cauchy-Riemann equation with good estimates hold under some geometrical conditions.

In this section $\mathbb{M} = (\mathbb{M}, H_{0,1}\mathbb{M})$ denotes an abstract compact CR manifold of class $\mathcal{C}^\infty$, of real dimension $2n-k$ and CR dimension $n-k$.

Let us define some anisotropic Sobolev spaces of functions:

- $\mathcal{S}^{0,p}(\mathbb{M})$, $1 < p < \infty$, is the set of L^p_{loc} functions on $\mathbb{M}$.
- $\mathcal{S}^{1,p}(\mathbb{M})$, $1 < p < \infty$, is the set of functions on $\mathbb{M}$ such that $f \in W^{\frac{1}{2},p}(\mathbb{M})$ and $X_\mathbb{C} f \in L^p_{\text{loc}}(\mathbb{M})$, for all complex tangent vector fields $X_\mathbb{C}$ to M.
- $\mathcal{S}^{l,p}(\mathbb{M})$, $l \geq 2$, $1 < p < \infty$, is the set of functions f such that $Xf \in \mathcal{S}^{l-2,p}(\mathbb{M})$, for all tangent vector fields X to M and $X_\mathbb{C} f \in \mathcal{S}^{l-1,p}(\mathbb{M})$, for all complex tangent vector fields $X_\mathbb{C}$ to M.

The sequence $(\mathcal{S}^{l,p}(\mathbb{M}),\ l \in \mathbb{N})$ is a sequence of anisotropic spaces in the sense of Section 2 with $\theta(l) = l+1$. Moreover $\cap_{l\in\mathbb{N}} \mathcal{S}^{l,p}(\mathbb{M}) = \mathcal{C}^\infty(\mathbb{M})$.

The anisotropic Hölder space of forms $\mathcal{S}^{l,p}_*(\mathbb{M})$, $l \geq 0$, $1 < p < \infty$, is then the space of forms on $\mathbb{M}$, whose coefficients are in $\mathcal{S}^{l,p}(\mathbb{M})$.

We have now to see if the sequence $(\mathcal{S}^{l,p}(\mathbb{M}),\ l \in \mathbb{N})$ is adapted to the $\overline{\partial}_b$-equation for some degree r.

The L^2 theory for $\Box_b$ in abstract CR manifolds of arbitrary codimension is developed in [9]. There it is proved that if $\mathbb{M}$ satisfies condition $Y(r)$ the Hodge decomposition theorem holds in degree r, which means that there exist a compact operator $N_b : L^2_{p,r}(\mathbb{M}) \to \mathrm{Dom}(\Box_b)$ and a continuous operator $H_b : L^2_{p,r}(\mathbb{M}) \to L^2_{p,r}(\mathbb{M})$ such that for any $f \in L^2_{p,r}(\mathbb{M})$

$$f = \overline{\partial}_b \overline{\partial}_b^* N_b f + \overline{\partial}_b^* \overline{\partial}_b N_b f + H_b f. \tag{3.1}$$

Moreover H_b vanishes on exact forms and if N_b is also defined on $L^2_{p,r+1}(\mathbb{M})$ then $N_b \overline{\partial}_b = \overline{\partial}_b N_b$.

Therefore if $\mathbb{M}$ satisfy both conditions $Y(r)$ and $Y(r+1)$ then (3.1) becomes an homotopy formula and using the Sobolev and the anisotropic Sobolev estimates in [9] (Theorems 3.3 and Corollary 1.3 (2)) we get the following result:

Proposition 3.1. *If $\mathbb{M}$ is q-concave, the sequence $(\mathcal{S}^{l,p}(\mathbb{M}),\ l \in \mathbb{N})$ of anisotropic spaces is adapted to the $\overline{\partial}_b$-equation in degree r for $0 \leq r \leq q-2$ and $n-k-q+1 \leq r \leq n-k$.*

Let us define now some anisotropic Hölder spaces of functions:

- $\mathcal{A}^\alpha(\mathbb{M})$, $0 < \alpha < 1$, is the set of continuous functions on $\mathbb{M}$ which are in $\mathcal{C}^{\alpha/2}(\mathbb{M})$.
- $\mathcal{A}^{1+\alpha}(\mathbb{M})$, $0 < \alpha < 1$, is the set of functions f such that $f \in \mathcal{C}^{(1+\alpha)/2}(\mathbb{M})$ and $X_\mathbb{C} f \in \mathcal{C}^{\alpha/2}(\mathbb{M})$, for all complex tangent vector fields $X_\mathbb{C}$ to $\mathbb{M}$. Set

$$\|f\|_{A\alpha} = \|f\|_{(1+\alpha)/2} + \sup_{\|X_\mathbb{C}\| \leq 1} \|X_\mathbb{C} f\|_{\alpha/2}. \tag{3.2}$$

- $\mathcal{A}^{l+\alpha}(\mathbb{M})$, $l \geq 2$, $0 < \alpha < 1$, is the set of functions f of class $\mathcal{C}^{[l/2]}$ such that $Xf \in \mathcal{A}^{l-2+\alpha}(\mathbb{M})$, for all tangent vector fields X to M and $X_\mathbb{C} f \in \mathcal{A}^{l-1+\alpha}(\mathbb{M})$, for all complex tangent vector fields $X_\mathbb{C}$ to $\mathbb{M}$.

The sequence $(\mathcal{A}^{l+\alpha}(\mathbb{M}),\ l \in \mathbb{N})$ is a sequence of anisotropic spaces in the sense of Section 2 with $\theta(l) = l + 1$. Moreover $\cap_{l\in\mathbb{N}}\mathcal{A}^{l+\alpha}(\mathbb{M}) = \mathcal{C}^\infty(\mathbb{M})$.

The anisotropic Hölder space of forms $\mathcal{A}_*^{l+\alpha}(\mathbb{M})$, $l \geq 0$, $0 < \alpha < 1$, is then the space of continuous forms on $\mathbb{M}$, whose coefficients are in $\mathcal{A}^{l+\alpha}(\mathbb{M})$.

It remains to see if the sequence $(\mathcal{A}^{l+\alpha}(\mathbb{M}),\ l \in \mathbb{N})$ is adapted to the $\overline{\partial}_b$-equation for some degrees r.

Assume $\mathbb{M}$ is locally embeddable and 1-concave. Then, by Proposition 3.1 in [4], there exist a complex manifold X and a smooth generic embedding $\mathcal{E} : \mathbb{M} \to M \subset X$ such that M is a smooth compact CR submanifold of X with the CR structure $H_{0,1}M = d\mathcal{E}(H_{0,1}\mathbb{M}) = T_{\mathbb{C}}M \cap T_{0,1}$. If E is a CR vector bundle over $\mathbb{M}$, by the 1-concavity of $\mathbb{M}$ and after an identification between $\mathbb{M}$ and M, the CR bundle E can be extended to an holomorphic bundle in a neighborhood of M, which we still denote by E. With these notations it follows from [5] that if $\mathbb{M}$ is q-concave, $q \geq 1$, there exist finite-dimensional subspaces $\mathcal{H}_r$ of $\mathcal{Z}^\infty_{n,r}(\mathbb{M}, E)$, $0 \leq r \leq q-1$ and $n-k-q+1 \leq r \leq n-k$, where $\mathcal{H}_0 = \mathcal{Z}^\infty_{n,0}(\mathbb{M}, E)$, continuous linear operators

$$A_r : \mathcal{C}^0_{n,r}(\mathbb{M}, E) \to \mathcal{C}^0_{n,r-1}(\mathbb{M}, E), \quad 1 \leq r \leq q \text{ and } n-k-q+1 \leq r \leq n-k$$

and continuous linear projections

$$P_r : \mathcal{C}^0_{n,r}(\mathbb{M}, E) \to \mathcal{C}^0_{n,r}(\mathbb{M}, E)\,, \qquad 0 \leq r \leq q-1 \text{ and } n-k-q+1 \leq r \leq n-k\,,$$

with

$$\operatorname{Im} P_r = \mathcal{H}_r\,, \quad 0 \leq r \leq q-1 \text{ and } n-k-q+1 \leq r \leq n-k\,, \tag{3.3}$$

and

$$\mathcal{C}^0_{n,r}(M, E)\cap\overline{\partial}_b\mathcal{C}^0_{n,r-1}(M, E) \subseteq \operatorname{Ker} P_r\,, \quad 1 \leq r \leq q-1 \text{ and } n-k-q+1 \leq r \leq n-k, \tag{3.4}$$

such that:

(i) For all $l \in \mathbb{N}$ and $1 \leq r \leq q$ or $n-k-q+1 \leq r \leq n-k$,

$$A_r(\mathcal{A}^{l+\alpha}_{n,r}(\mathbb{M}, E)) \subset \mathcal{A}^{l+1\alpha}_{n,r-1}(\mathbb{M}, E)$$

and A_r is continuous as an operator between $\mathcal{A}^{l+\alpha}_{n,r}(\mathbb{M}, E)$ and $\mathcal{A}^{l+1+\alpha}_{n,r-1}(\mathbb{M}, E)$.

(ii) For all $0 \leq r \leq q-1$ or $n-k-q+1 \leq r \leq n-k$ and $f \in \mathcal{C}^0_{n,r}(\mathbb{M}, E)$ with $\overline{\partial}_b f \in \mathcal{C}^0_{n,r+1}(\mathbb{M}, E)$,

$$f - P_r f = \begin{cases} A_1\overline{\partial}_b f & \text{if } r = 0\,, \\ \overline{\partial}_b A_r f + A_{r+1}\overline{\partial}_b f & \text{if } 1 \leq r \leq q-1 \text{ or } n-k-q+1 \leq r \leq n-k\,. \end{cases} \tag{3.5}$$

This implies the following result

Proposition 3.2. *If $\mathbb{M}$ is locally embeddable and q-concave the sequence $\mathcal{A}^{l+\alpha}(\mathbb{M})$, $l \in \mathbb{N}$ of anisotropic spaces is adapted to the $\overline{\partial}_b$-equation in degree r for $1 \leq r \leq q-1$ and $n-k-q+1 \leq r \leq n-k$*

Finally let us recall the definition of the anisotropic Hölder spaces $\Gamma^{l+\alpha}(\mathbb{M})$ of Folland and Stein.

– $\Gamma^{\alpha}(\mathbb{M})$, $0<\alpha<1$, is the set of continuous functions in $\mathbb{M}$ such that if for every $x_0\in\mathbb{M}$
$$\sup_{\gamma(.)}\frac{|f(\gamma(t)-f(x_0)|}{|t|^{\alpha}}<\infty$$
for any complex tangent curve γ through x_0.

– $\Gamma^{l+\alpha}(\mathbb{M})$, $l\geq 1$, $0<\alpha<1$, is the set of continuous functions in $\mathbb{M}$ such that $X_{\mathbb{C}}f\in\Gamma^{l-1+\alpha}(\mathbb{M})$, for all complex tangent vector fields $X_{\mathbb{C}}$ to $\mathbb{M}$.

The spaces $\Gamma^{l+\alpha}(\mathbb{M})$ are subspaces of the spaces $\mathcal{A}^{l+\alpha}(\mathbb{M})$.

Note that by Corollary 1.3 (1) in [9] and Section 3 in [5], Propositions 3.1 and 3.2 hold also for the anisotropic Hölder spaces $\Gamma^{l+\alpha}(\mathbb{M})$ of Folland and Stein.

Let us summarize all this in connection with Section 2 in the next theorem.

Theorem 3.3. *If* $\mathbb{M}$ *is* q*-concave,*

(i) *Theorem* 2.1 *holds for* $\mathcal{B}^l(\mathbb{M})=\mathcal{S}^{l,p}(\mathbb{M})$ *with* $s_1(q)=q-2$ *and* $s_2(q)=n-k-q+1$ *in the abstract case,*

(ii) *Theorem* 2.1 *holds for* $\mathcal{B}^l(\mathbb{M})=\mathcal{A}^{l+\alpha}(\mathbb{M})$ *with* $s_1(q)=q-1$ *and* $s_2(q)=n-k-q+1$ *when* $\mathbb{M}$ *is locally embeddable*

(iii) *Theorem* 2.1 *holds for* $\mathcal{B}^l(\mathbb{M})=\Gamma^{l+\alpha}(\mathbb{M})$ *with* $s_1(q)=q-2$ *in the abstract case and* $s_1(q)=q-1$ *when* $\mathbb{M}$ *is locally embeddable, and with* $s_2(q)=n-k-q+1$ *in both cases.*

Since in all the three cases of Theorem 3.3 we have $\cap_{l\in\mathbb{N}}B_l(\mathbb{M})=\mathcal{C}^{\infty}(\mathbb{M})$, Corollary 2.3 becomes

Corollary 3.4. *Let* $\mathbb{M}=(\mathbb{M},H_{0,1}\mathbb{M})$ *be an abstract compact CR manifold of class* $\mathcal{C}^{\infty}$*, of real dimension* $2n-k$ *and CR dimension* $n-k$*, and* $\widehat{\mathbb{M}}=(\mathbb{M},\widehat{H}_{0,1}\mathbb{M})$ *another abstract compact CR manifold such that* $\widehat{H}_{0,1}\mathbb{M}$ *is an integrable horizontal smooth perturbation of* $H_{0,1}\mathbb{M}$*. Let* $\Phi\in\mathcal{C}^{\infty}_{0,1}(\mathbb{M},H_{1,0}\mathbb{M})$ *be the differential form which defines the tangential Cauchy-Riemann operator* $\overline{\partial}_b^{\Phi}=\overline{\partial}_b-\Phi\lrcorner\partial_b$ *associated to the CR structure* $\widehat{H}_{0,1}\mathbb{M}$*. Assume* $\mathbb{M}$ *is* q*-concave, then there exists a sequence* $(\delta_l)_{l\in\mathbb{N}}$ *of positive real numbers such that, if* $\|\Phi\|_l<\delta_l$ *for each* $l\in\mathbb{N}$

(i) $H^{p,r-p}(\mathbb{M})=0$*, for all* $1\leq p\leq r$*, implies* $H^{0,r}_{\Phi}(\widehat{\mathbb{M}})=0$*, when* $1\leq r\leq q-2$ *in the abstract case and also for* $r=q-1$ *if* $\mathbb{M}$ *is locally embeddable,*

(ii) $H^{n-p,r+p}(\mathbb{M})=0$*, for all* $0\leq p\leq n-k-r$*, implies* $H^{n,r}_{\Phi}(\widehat{\mathbb{M}})=0$*, when* $n-k-q+1\leq r\leq n-k$*.*

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Coherent Sheaves and Cohesive Sheaves

László Lempert

To Linda Rothschild on her birthday

Abstract. We consider coherent and cohesive sheaves of $\mathcal{O}$-modules over open sets $\Omega \subset \mathbb{C}^n$. We prove that coherent sheaves, and certain other sheaves derived from them, are cohesive; and conversely, certain sheaves derived from cohesive sheaves are coherent. An important tool in all this, also proved here, is that the sheaf of Banach space valued holomorphic germs is flat.

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1. Introduction

The theory of coherent sheaves has been central to algebraic and analytic geometry in the past fifty years. By contrast, in infinite-dimensional analytic geometry coherence is irrelevant, as most sheaves associated with infinite-dimensional complex manifolds are not even finitely generated over the structure sheaf, let alone coherent. In a recent paper with Patyi, [LP], we introduced the class of so-called cohesive sheaves in Banach spaces, that seems to be the correct replacement of coherent sheaves – we were certainly able to show that many sheaves that occur in the subject are cohesive, and for cohesive sheaves Cartan's Theorems A and B hold. We will go over the definition of cohesive sheaves in Section 2, but for a precise formulation of the results above the reader is advised to consult [LP].

While cohesive sheaves were designed to deal with infinite-dimensional problems, they make sense in finite-dimensional spaces as well, and there are reasons to study them in this context, too. First, some natural sheaves even over finite-dimensional manifolds are not finitely generated: for example the sheaf $\mathcal{O}^E$ of germs of holomorphic functions taking values in a fixed infinite-dimensional Banach space E is not. It is not quasicoherent, either (for this notion, see [Ha]), but

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it is cohesive. Second, a natural approach to study cohesive sheaves in infinite-dimensional manifolds would be to restrict them to various finite-dimensional submanifolds.

The issue to be addressed in this paper is the relationship between coherence and cohesion in finite-dimensional spaces. Our main results are Theorems 4.3, 4.4, and 4.1. Loosely speaking, the first says that coherent sheaves are cohesive, and the second that they remain cohesive even after tensoring with the sheaf $\mathcal{O}^F$ of holomorphic germs valued in a Banach space F. A key element of the proof is that $\mathcal{O}^F$ is flat, Theorem 4.1. This latter is also relevant for the study of subvarieties. On the other hand, Masagutov showed that $\mathcal{O}^F$ is not free in general, see [Ms, Corollary 1.4].

The results above suggest two problems, whose resolution has eluded us. First, is the tensor product of a coherent sheaf with a cohesive sheaf itself cohesive? Of course, one can also ask the more ambitious question whether the tensor product of two cohesive sheaves is cohesive, but here one should definitely consider some kind of "completed" tensor product, and it is part of the problem to find which one. The second problem is whether a finitely generated cohesive sheaf is coherent. If so, then coherent sheaves could be defined as cohesive sheaves of finite type. We could only solve some related problems: according to Corollary 4.2, any finitely generated subsheaf of $\mathcal{O}^F$ is coherent; and cohesive subsheaves of coherent sheaves are also coherent, Theorem 5.4.

2. Cohesive sheaves, an overview

In this Section we will review notions and theorems related to the theory of cohesive sheaves, following [LP]. We assume the reader is familiar with very basic sheaf theory. One good reference to what we need here – and much more – is [S]. Let $\Omega \subset \mathbb{C}^n$ be an open set and E a complex Banach space. A function $f\colon \Omega \to E$ is holomorphic if for each $a \in \Omega$ there is a linear map $L\colon \mathbb{C}^n \to E$ such that

$$f(z) = f(a) + L(z-a) + o\|z-a\|, \qquad z \to a.$$

This is equivalent to requiring that in each ball $B \subset \Omega$ centered at any $a \in \Omega$ our f can be represented as a locally uniformly convergent power series $f(z) = \sum_j c_j(z-a)^j$, with $j = (j_1, \dots, j_n)$ a nonnegative multiindex and $c_j \in E$. We denote by $\mathcal{O}^E_\Omega$ or just $\mathcal{O}^E$ the sheaf of holomorphic E-valued germs over Ω. In particular, $\mathcal{O} = \mathcal{O}^{\mathbb{C}}$ is a sheaf of rings, and $\mathcal{O}^E$ is a sheaf of $\mathcal{O}$-modules. Typically, instead of a sheaf of $\mathcal{O}$-modules we will just talk about $\mathcal{O}$-modules.

Definition 2.1. The sheaves $\mathcal{O}^E = \mathcal{O}^E_\Omega \to \Omega$ are called plain sheaves.

Theorem 2.2 ([Bi, Theorem 4], [Bu, p. 331] or [L, Theorem 2.3]). *If $\Omega \subset \mathbb{C}^n$ is pseudoconvex and $q = 1, 2, \dots,$ then $H^q(\Omega, \mathcal{O}^E) = 0$.*

Given another Banach space F, we write $\mathrm{Hom}(E, F)$ for the Banach space of continuous linear maps $E \to F$. If $U \subset \Omega$ is open, then any holomorphic $\Phi\colon U \to$

$\mathrm{Hom}(E,F)$ induces a homomorphism $\varphi\colon \mathcal{O}^E|U \to \mathcal{O}^F|U$, by associating with the germ of a holomorphic $e\colon V \to E$ at $\zeta \in V \subset U$ the germ of the function $z \mapsto \Phi(z)e(z)$, again at ζ. Such homomorphisms and their germs are called plain. The sheaf of plain homomorphisms between $\mathcal{O}^E$ and $\mathcal{O}^F$ is denoted $\mathbf{Hom}_{\mathrm{plain}}(\mathcal{O}^E,\mathcal{O}^F)$. If $\mathbf{Hom}_{\mathcal{O}}(\mathcal{A},\mathcal{B})$ denotes the sheaf of $\mathcal{O}$-homomorphisms between $\mathcal{O}$-modules $\mathcal{A}$ and $\mathcal{B}$, then

$$\mathbf{Hom}_{\mathrm{plain}}(\mathcal{O}^E,\mathcal{O}^F) \subset \mathbf{Hom}_{\mathcal{O}}(\mathcal{O}^E,\mathcal{O}^F) \tag{2.1}$$

is an $\mathcal{O}$-submodule. In fact, Masagutov showed that the two sides in (2.1) are equal unless $n = 0$, see [Ms, Theorem 1.1], but for the moment we do not need this. The $\mathcal{O}(U)$-module of sections $\Gamma(U,\mathbf{Hom}_{\mathrm{plain}}(\mathcal{O}^E,\mathcal{O}^F))$ is in one-to-one correspondence with the $\mathcal{O}(U)$-module $\mathrm{Hom}_{\mathrm{plain}}(\mathcal{O}^E|U,\mathcal{O}^F|U)$ of plain homomorphisms. Further, any germ $\Phi \in \mathcal{O}_z^{\mathrm{Hom}(E,F)}$ induces a germ $\varphi \in \mathbf{Hom}_{\mathrm{plain}}(\mathcal{O}^E,\mathcal{O}^F)_z$. As pointed out in [LP, Section 2], the resulting map is an isomorphism

$$\mathcal{O}^{\mathrm{Hom}(E,F)} \overset{\approx}{\to} \mathbf{Hom}_{\mathrm{plain}}(\mathcal{O}^E,\mathcal{O}^F) \tag{2.2}$$

of $\mathcal{O}$-modules.

Definition 2.3. An analytic structure on an $\mathcal{O}$-module $\mathcal{A}$ is the choice, for each plain sheaf $\mathcal{E}$, of a submodule $\mathbf{Hom}(\mathcal{E},\mathcal{A}) \subset \mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{A})$, subject to

(i) if $\mathcal{E},\mathcal{F}$ are plain sheaves and $\varphi \in \mathbf{Hom}_{\mathrm{plain}}(\mathcal{E},\mathcal{F})_z$ for some $z \in \Omega$, then $\varphi^*\mathbf{Hom}(\mathcal{F},\mathcal{A})_z \subset \mathbf{Hom}(\mathcal{E},\mathcal{A})_z$; and
(ii) $\mathbf{Hom}(\mathcal{O},\mathcal{A}) = \mathbf{Hom}_{\mathcal{O}}(\mathcal{O},\mathcal{A})$.

If $\mathcal{A}$ is endowed with an analytic structure, one says that $\mathcal{A}$ is an analytic sheaf. The reader will realize that this is different from the traditional terminology, where "analytic sheaves" and "$\mathcal{O}$-modules" mean one and the same thing.

For example, one can endow a plain sheaf $\mathcal{G}$ with an analytic structure by setting

$$\mathbf{Hom}(\mathcal{E},\mathcal{G}) = \mathbf{Hom}_{\mathrm{plain}}(\mathcal{E},\mathcal{G}).$$

Unless stated otherwise, we will always consider plain sheaves endowed with this analytic structure. – Any $\mathcal{O}$-module $\mathcal{A}$ has two extremal analytic structures. The maximal one is given by $\mathbf{Hom}(\mathcal{E},\mathcal{A}) = \mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{A})$. In the minimal structure, $\mathbf{Hom}_{\mathrm{min}}(\mathcal{E},\mathcal{A})$ consists of germs α that can be written $\alpha = \sum \beta_j\gamma_j$ with

$$\gamma_j \in \mathbf{Hom}_{\mathrm{plain}}(\mathcal{E},\mathcal{O}) \quad \text{and} \quad \beta_j \in \mathbf{Hom}_{\mathcal{O}}(\mathcal{O},\mathcal{A}), \quad j = 1,\dots,k.$$

An $\mathcal{O}$-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ of $\mathcal{O}$-modules induces a homomorphism

$$\varphi_*\colon \mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{A}) \to \mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{B})$$

for $\mathcal{E}$ plain. When $\mathcal{A}$, $\mathcal{B}$ are analytic sheaves, we say that φ is analytic if

$$\varphi_*\mathbf{Hom}(\mathcal{E},\mathcal{A}) \subset \mathbf{Hom}(\mathcal{E},\mathcal{B})$$

for all plain sheaves $\mathcal{E}$. It is straightforward to check that if $\mathcal{A}$ and $\mathcal{B}$ themselves are plain sheaves, then φ is analytic precisely when it is plain. We write $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ for the $\mathcal{O}(\Omega)$-module of analytic homomorphisms $\mathcal{A} \to \mathcal{B}$ and $\mathbf{Hom}(\mathcal{A},\mathcal{B})$ for the sheaf of germs of analytic homomorphisms $\mathcal{A}|U \to \mathcal{B}|U$, with $U \subset \Omega$ open. Again, one easily checks that, when $\mathcal{A} = \mathcal{E}$ is plain, this new notation is consistent with the one already in use. Further,

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \approx \Gamma(\Omega, \mathbf{Hom}(\mathcal{A}, \mathcal{B})). \tag{2.3}$$

Definition 2.4. Given an $\mathcal{O}$-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of $\mathcal{O}$-modules, any analytic structure on $\mathcal{B}$ induces one on $\mathcal{A}$ by the formula

$$\mathbf{Hom}(\mathcal{E}, \mathcal{A}) = \varphi_*^{-1}\mathbf{Hom}(\mathcal{E}, \mathcal{B}).$$

If φ is an epimorphism, then any analytic structure on $\mathcal{A}$ induces one on $\mathcal{B}$ by the formula

$$\mathbf{Hom}(\mathcal{E}, \mathcal{B}) = \varphi_*\mathbf{Hom}(\mathcal{E}, \mathcal{A}).$$

[LP, 3.4] explains this construction in the cases when φ is the inclusion of a submodule $\mathcal{A} \subset \mathcal{B}$ and when φ is the projection on a quotient $\mathcal{B} = \mathcal{A}/\mathcal{C}$.

Given a family $\mathcal{A}_i$, $i \in I$, of analytic sheaves, an analytic structure is induced on the sum $\mathcal{A} = \bigoplus \mathcal{A}_i$. For any plain $\mathcal{E}$ there is a natural homomorphism

$$\bigoplus_i \mathbf{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{A}_i) \to \mathbf{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{A}),$$

and we define the analytic structure on $\mathcal{A}$ by letting $\mathbf{Hom}(\mathcal{E}, \mathcal{A})$ be the image of $\bigoplus \mathbf{Hom}(\mathcal{E}, \mathcal{A}_i)$. With this definition, the inclusion maps $\mathcal{A}_i \to \mathcal{A}$ and the projections $\mathcal{A} \to \mathcal{A}_i$ are analytic.

Definition 2.5. A sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of analytic sheaves and homomorphisms over Ω is said to be completely exact if for every plain sheaf $\mathcal{E}$ and every pseudoconvex $U \subset \Omega$ the induced sequence

$$\mathrm{Hom}(\mathcal{E}|U, \mathcal{A}|U) \to \mathrm{Hom}(\mathcal{E}|U, \mathcal{B}|U) \to \mathrm{Hom}(\mathcal{E}|U, \mathcal{C}|U)$$

is exact. A general sequence of analytic homomorphisms is completely exact if every three-term subsequence is completely exact.

Definition 2.6. An infinite completely exact sequence

$$\cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{A} \to 0 \tag{2.4}$$

of analytic homomorphisms is called a complete resolution of $\mathcal{A}$ if each $\mathcal{F}_j$ is plain.

When Ω is finite dimensional, as in this paper, complete resolutions can be defined more simply:

Theorem 2.7. *Let*

$$\cdots \to \mathcal{F}_2 \xrightarrow{\varphi_2} \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{A} \xrightarrow{\varphi_0} 0 \tag{2.5}$$

be an infinite sequence of analytic homomorphisms over $\Omega \subset \mathbb{C}^n$, with each $\mathcal{F}_j$ plain. If for each plain $\mathcal{E}$ over Ω the induced sequence

$$\cdots \to \mathbf{Hom}(\mathcal{E}, \mathcal{F}_2) \to \mathbf{Hom}(\mathcal{E}, \mathcal{F}_1) \to \mathbf{Hom}(\mathcal{E}, \mathcal{A}) \to 0 \tag{2.6}$$

is exact, then (2.5) *is completely exact.*

Proof. Setting $\mathcal{E} = \mathcal{O}$ in (2.6) we see that (2.5) is exact. Let $\mathcal{K}_j = \operatorname{Ker} \varphi_j = \operatorname{Im} \varphi_{j+1}$, and endow it with the analytic structure induced by the embedding $\mathcal{K}_j \hookrightarrow \mathcal{F}_j$, as in Definition 2.4. The exact sequence $\mathcal{O} \to \mathcal{K}_j \hookrightarrow \mathcal{F}_j \xrightarrow{\varphi_j} \mathcal{K}_{j-1} \to 0$ induces a sequence

$$0 \to \mathbf{Hom}(\mathcal{E}, \mathcal{K}_j) \to \mathbf{Hom}(\mathcal{E}, \mathcal{F}_j) \to \mathbf{Hom}(\mathcal{E}, \mathcal{K}_{j-1}) \to 0, \tag{2.7}$$

also exact since (2.6) was. Let $U \subset \Omega$ be pseudoconvex. Then in the long exact sequence associated with (2.7)

$$\cdots \to H^q(U, \mathbf{Hom}(\mathcal{E}, \mathcal{F}_j)) \to H^q(U, \mathbf{Hom}(\mathcal{E}, \mathcal{K}_{j-1})) \to \\ \to H^{q+1}(U, \mathbf{Hom}(\mathcal{E}, \mathcal{K}_j)) \to H^{q+1}(U, \mathbf{Hom}(\mathcal{E}, \mathcal{F}_j)) \to \cdots \tag{2.8}$$

the first and last terms indicated vanish for $q \geq 1$ by virtue of Theorem 2.2 and (2.2). Hence the middle terms are isomorphic:

$$H^q(U, \mathbf{Hom}(\mathcal{E}, \mathcal{K}_{j-1})) \approx H^{q+1}(U, \mathbf{Hom}(\mathcal{E}, \mathcal{K}_j)) \approx \cdots \\ \cdots \approx H^{q+n}(U, \mathbf{Hom}(\mathcal{E}, \mathcal{K}_{j+n-1})) \approx 0.$$

Using this and (2.3), the first few terms of the sequence (2.8) are

$$0 \to \operatorname{Hom}(\mathcal{E}|U, \mathcal{K}_j|U) \to \operatorname{Hom}(\mathcal{E}|U, \mathcal{F}_j|U) \to \operatorname{Hom}(\mathcal{E}|U, \mathcal{K}_{j-1}|U) \to 0.$$

The exactness of this latter implies $\cdots \to \operatorname{Hom}(\mathcal{E}|U, \mathcal{F}_1|U) \to \operatorname{Hom}(\mathcal{E}|U, \mathcal{A}|U) \to 0$ is exact, and so (2.5) is indeed completely exact. □

Definition 2.8. An analytic sheaf $\mathcal{A}$ over $\Omega \subset \mathbb{C}^n$ is cohesive if each $z \in \Omega$ has a neighborhood over which $\mathcal{A}$ has a complete resolution.

The simplest examples of cohesive sheaves are the plain sheaves, that have complete resolutions of form $\cdots \to 0 \to 0 \to \mathcal{E} \to \mathcal{E} \to 0$. The main result of [LP] implies the following generalization of Cartan's Theorems A and B, see Theorem 2 of the Introduction there:

Theorem 2.9. *Let $\mathcal{A}$ be a cohesive sheaf over a pseudoconvex $\Omega \subset \mathbb{C}^n$. Then*
(a) *$\mathcal{A}$ has a complete resolution over all of Ω;*
(b) *$H^q(\Omega, \mathcal{A}) = 0$ for $q \geq 1$.*

3. Tensor products

Let R be a commutative ring with a unit and A, B two R-modules. Recall that the tensor product $A \otimes_R B = A \otimes B$ is the R-module freely generated by the set $A \times B$, modulo the submodule generated by elements of form

$$(ra + a', b) - r(a, b) - (a', b) \quad \text{and} \quad (a, rb + b') - r(a, b) - (a, b'),$$

where $r \in R$, $a, a' \in A$, and $b, b' \in B$. The class of $(a, b) \in A \times B$ in $A \otimes B$ is denoted $a \otimes b$. Given homomorphisms $\alpha\colon A \to A'$, $\beta\colon B \to B'$ of R-modules, $\alpha \otimes \beta\colon A \otimes B \to A' \otimes B'$ denotes the unique homomorphism satisfying $(\alpha \otimes \beta)(a \otimes b) = \alpha(a) \otimes \beta(b)$.

A special case is the tensor product of Banach spaces A, B; here $R = \mathbb{C}$. The tensor product $A \otimes B$ is just a vector space, on which in general there are several natural ways to introduce a norm. However, when $\dim A = k < \infty$, all those norms are equivalent, and turn $A \otimes B$ into a Banach space. For example, if a basis $a_1, \dots, a_k$ of A is fixed, any $v \in A \otimes B$ can be uniquely written $v = \sum a_j \otimes b_j$, with $b_j \in B$. Then $A \otimes B$ with the norm

$$\|v\| = \max_j \|b_j\|_B$$

is isomorphic to $B^{\oplus k}$.

Similarly, if $\mathcal{R}$ is a sheaf of commutative unital rings over a topological space Ω and $\mathcal{A}, \mathcal{B}$ are $\mathcal{R}$-modules, then the tensor product sheaf $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$ can be defined, see, e.g., [S]. The tensor product is itself a sheaf of $\mathcal{R}$-modules, its stalks $(\mathcal{A} \otimes \mathcal{B})_x$ are just the tensor products of $\mathcal{A}_x$ and $\mathcal{B}_x$ over $\mathcal{R}_x$. Fix now an open $\Omega \subset \mathbb{C}^n$, an $\mathcal{O}$-module $\mathcal{A}$, and an analytic sheaf $\mathcal{B}$ over Ω. An analytic structure can be defined on $\mathcal{A} \otimes \mathcal{B}$ as follows. For any plain sheaf $\mathcal{E}$ there is a tautological $\mathcal{O}$-homomorphism

$$T = T_{\mathcal{E}} \colon \mathcal{A} \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B}) \to \mathbf{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{A} \otimes \mathcal{B}), \tag{3.1}$$

obtained by associating with $a \in \mathcal{A}_\zeta$, $\epsilon \in \mathbf{Hom}(\mathcal{E}, \mathcal{B})_\zeta$ first a section $\tilde{a}$ of $\mathcal{A}$ over a neighborhood U of ζ, such that $\tilde{a}(\zeta) = a$; then defining $\tau^a \in \mathbf{Hom}_{\mathcal{O}}(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})_\zeta$ as the germ of the homomorphism

$$\mathcal{B}_z \ni b \mapsto \tilde{a}(z) \otimes b \in \mathcal{A}_z \otimes \mathcal{B}_z, \qquad z \in U;$$

and finally letting $T(a \otimes \epsilon) = \tau^a \epsilon$.

Definition 3.1. The (tensor product) analytic structure on $\mathcal{A} \otimes \mathcal{B}$ is given by $\mathbf{Hom}(\mathcal{E}, \mathcal{A} \otimes \mathcal{B}) = \operatorname{Im} T_{\mathcal{E}}$.

One quickly checks that this prescription indeed satisfies the axioms of an analytic structure. Equivalently, one can define $\mathbf{Hom}(\mathcal{E}, \mathcal{A} \otimes \mathcal{B}) \subset \mathbf{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{A} \otimes \mathcal{B})$ as the submodule spanned by germs of homomorphisms of the form

$$\mathcal{E}|U \xrightarrow{\approx} \mathcal{O} \otimes \mathcal{E}|U \xrightarrow{\alpha \otimes \beta} \mathcal{A} \otimes \mathcal{B}|U,$$

where $U \subset \Omega$ is open, $\alpha : \mathcal{O}|U \to \mathcal{A}|U$ and $\beta : \mathcal{E}|U \to \mathcal{B}|U$ are $\mathcal{O}$-, resp. analytic homomorphisms (and the first isomorphism is the canonical one). The following is obvious.

Proposition 3.2. *T in* (3.1) *is natural: if $\alpha \colon \mathcal{A} \to \mathcal{A}'$ and $\beta \colon \mathcal{B} \to \mathcal{B}'$ are $\mathcal{O}$-, resp. analytic homomorphisms, then T and the corresponding T' fit in a commutative diagram*

$$\begin{array}{ccc} \mathcal{A} \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B}) & \xrightarrow{\alpha \otimes \beta_*} & \mathcal{A}' \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B}') \\ \downarrow T & & \downarrow T' \\ \mathbf{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{A} \otimes \mathcal{B}) & \xrightarrow{(\alpha \otimes \beta)_*} & \mathbf{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{A}' \otimes \mathcal{B}'). \end{array}$$

Corollary 3.3. *If α, β are as above, then $\alpha \otimes \beta\colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \otimes \mathcal{B}'$ is analytic.*

Proposition 3.4. *If $\mathcal{A}$ is an $\mathcal{O}$-module and $\mathcal{B}$ an analytic sheaf, then the tensor product analytic structure on $\mathcal{A} \otimes \mathcal{O}$ is the minimal one. Further, the map*

$$\mathcal{B} \ni b \mapsto 1 \otimes b \in \mathcal{O} \otimes \mathcal{B}$$

is an analytic isomorphism.

Both statements follow from inspecting the definitions.

Proposition 3.5. *If $\mathcal{A}, \mathcal{A}_i$ are $\mathcal{O}$-modules and $\mathcal{B}, \mathcal{B}_i$ are analytic sheaves, then the obvious $\mathcal{O}$-isomorphisms*

$$(\bigoplus \mathcal{A}_i) \otimes \mathcal{B} \xrightarrow{\approx} \bigoplus (\mathcal{A}_i \otimes \mathcal{B}), \quad \mathcal{A} \otimes (\bigoplus \mathcal{B}_i) \xrightarrow{\approx} \bigoplus (\mathcal{A} \otimes \mathcal{B}_i)$$

are in fact analytic isomorphisms.

This follows from Definition 3.1, upon taking into account the distributive property of the tensor product of $\mathcal{O}$-modules. Consider now a finitely generated plain sheaf $\mathcal{F} = \mathcal{O}^F \approx \mathcal{O} \oplus \cdots \oplus \mathcal{O}$, with $\dim F = k$. By putting together Propositions 3.4 and 3.5 we obtain analytic isomorphisms

$$\mathcal{F} \otimes \mathcal{B} \approx (\mathcal{O} \otimes \mathcal{B}) \oplus \cdots \oplus (\mathcal{O} \otimes \mathcal{B}) \approx \mathcal{B} \oplus \cdots \oplus \mathcal{B}.$$

When $\mathcal{B} = \mathcal{O}^B$ is plain, this specializes to

$$\mathcal{O}^F \otimes \mathcal{O}^B \approx \mathcal{O}^B \oplus \cdots \oplus \mathcal{O}^B \approx \mathcal{O}^{B^{\oplus k}} \approx \mathcal{O}^{F \otimes B}. \tag{3.2}$$

Later on we will need to know that inducing, in the sense of Definition 2.4, and tensoring are compatible. Here we discuss the easy case, an immediate consequence of the tensor product being a right exact functor; the difficult case will have to wait until Section 6.

Proposition 3.6. *Let $\psi\colon \mathcal{A} \to \mathcal{A}'$ be an epimorphism of $\mathcal{O}$-modules and $\mathcal{B}$ an analytic sheaf. Then the tensor product analytic structure on $\mathcal{A}' \otimes \mathcal{B}$ is induced* (*in the sense of Definition* 2.4) *from the tensor product analytic structure on $\mathcal{A} \otimes \mathcal{B}$ by the epimorphism $\psi \otimes id_{\mathcal{B}}\colon \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}' \otimes \mathcal{B}$.*

Proof. We write $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A}' \otimes \mathcal{B}$ for the analytic sheaves endowed with the tensor product structure. The claim means

$$(\psi \otimes \mathrm{id}_{\mathcal{B}})_* \mathbf{Hom}(\mathcal{E}, \mathcal{A} \otimes \mathcal{B}) = \mathbf{Hom}(\mathcal{E}, \mathcal{A}' \otimes \mathcal{B})$$

for every plain $\mathcal{E}$. But this follows from Definition 3.1 if we take into account the naturality of T (Proposition 3.2) and that

$$\psi \otimes \mathrm{id}_{\mathbf{Hom}(\mathcal{E},\mathcal{B})}\colon \mathcal{A} \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B}) \to \mathcal{A}' \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B})$$

is onto. □

In the sequel it will be important to know when T in (3.1) is injective. This issue is somewhat subtle and depends on the analysis of Section 5.

4. The main results

We fix an open set $\Omega \subset \mathbb{C}^n$. In the remainder of this paper all sheaves, unless otherwise stated, will be over Ω.

Theorem 4.1. *Let $\mathcal{F}$ be a plain sheaf, $\mathcal{A} \subset \mathcal{F}$ finitely generated, and $\zeta \in \Omega$. Then on some open $U \ni \zeta$ there is a finitely generated free subsheaf $\mathcal{E} \subset \mathcal{F}|U$ that contains $\mathcal{A}|U$. In particular, plain sheaves are flat.*

Recall that an $\mathcal{O}$-module $\mathcal{F}$ is flat if for every exact sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of $\mathcal{O}$-modules the induced sequence $\mathcal{A} \otimes \mathcal{F} \to \mathcal{B} \otimes \mathcal{F} \to \mathcal{C} \otimes \mathcal{F}$ is exact.

Theorem 4.1 and Oka's coherence theorem imply

Corollary 4.2. *Finitely generated submodules of a plain sheaf are coherent.*

Theorem 4.3. *A coherent sheaf, endowed with its minimal analytic structure, is cohesive.*

Theorem 4.4. *If $\mathcal{A}$ is a coherent sheaf and $\mathcal{B}$ is a plain sheaf, then $\mathcal{A} \otimes \mathcal{B}$ is cohesive.*

Theorem 4.1 will be proved in Section 5, Theorems 4.3 and 4.4 in Section 7.

5. Preparation

The main result of this Section is the following. Throughout, $\Omega \subset \mathbb{C}^n$ will be open.

Lemma 5.1. *Let P, Q be Banach spaces, $f\colon \Omega \to \mathrm{Hom}(P,Q)$ holomorphic, and $\zeta \in \Omega$.*

(a) *If $\dim P < \infty$ then there are a finite-dimensional $Q' \subset Q$, an open $U \ni \zeta$, and a holomorphic $q\colon U \to \mathrm{GL}(Q)$ such that $\mathrm{Im}\, q(z)f(z) \subset Q'$ for all $z \in U$.*

(b) *If $\dim Q < \infty$ then there are a finite codimensional $P' \subset P$, an open $U \ni \zeta$, and a holomorphic $p\colon U \to \mathrm{GL}(P)$ such that $P' \subset \mathrm{Ker}\, f(z)p(z)$ for all $z \in U$.*

The proof depends on various extensions of the Weierstrass Preparation Theorem. That the road to coherence leads through the Preparation Theorem is, of course, an old idea of Oka. Let A be a Banach algebra with unit $\mathbf{1}$, and let $A^\times \subset A$ denote the open set of invertible elements.

Lemma 5.2. *Let $f\colon \Omega \to A$ be holomorphic, $0 \in \Omega$, and $d = 0, 1, 2, \ldots$ such that*

$$\frac{\partial^j f}{\partial z_1^j}(0) = 0 \quad for \quad j < d, \qquad and \quad \frac{\partial^d f}{\partial z_1^d}(0) \in A^\times.$$

Then on some open $U \ni 0$ there is a holomorphic $\Phi\colon U \to A^\times$ such that, writing $z = (z_1, z')$

$$\Phi(z)f(z) = \mathbf{1}z_1^d + \sum_{j=0}^{d-1} f_j(z')z_1^j, \quad z \in U, \tag{5.1}$$

and $f_j(0) = 0$.

We refer the reader to [Hö, 6.1]. The proof of Weierstrass' theorem given there for the case $A = \mathbb{C}$ applies in this general setting as well.

Lemma 5.3. *Let $0 \in \Omega$, E a Banach space, E^* its dual, $g\colon \Omega \to E$ (resp. $h\colon \Omega \to E^*$) holomorphic functions such that*

$$\frac{\partial^j g}{\partial z_1^j}(0) \neq 0 \qquad \left(\text{resp. } \frac{\partial^j h}{\partial z_1^j}(0) \neq 0\right), \quad \text{for some } j. \tag{5.2}$$

Then there are an open $U \ni 0$, a holomorphic $\Phi\colon U \to \mathrm{GL}(E)$, and $0 \neq e \in E$ (resp. $0 \neq e^* \in E^*$), such that

$$\begin{gathered}\Phi(z)g(z) = ez_1^d + \sum_{j=0}^{d-1} g_j(z')z_1^j \quad , \quad z \in U,\\ \left(\text{resp. } h(z)\Phi(z) = e^* z_1^d + \sum_{j=0}^{d-1} h_j(z')z_1^j\right),\end{gathered} \tag{5.3}$$

with some $d = 0, 1, \ldots$, and $g_j(0) = 0$, $h_j(0) = 0$.

Proof. We will only prove for g, the proof for h is similar. The smallest j for which (5.2) holds will be denoted d. Thus $\partial^d g/\partial z_1^d(0) = e \neq 0$. Let $V \subset E$ be a closed subspace complementary to the line spanned by e, and define a holomorphic $f\colon \Omega \to \mathrm{Hom}(E, E)$ by

$$f(z)(\lambda e + v) = \lambda g(z) + v z_1^d/d!, \quad \lambda \in \mathbb{C},\ v \in V.$$

We apply Lemma 5.2 with the Banach algebra $A = \mathrm{Hom}(E, E)$; its invertibles form $A^\times = \mathrm{GL}(E)$. As

$$\frac{\partial^j f}{\partial z_1^j}(0) = 0 \text{ for } j < d \quad \text{and} \quad \frac{\partial^d f}{\partial z_1^d}(0) = \mathrm{id}_E,$$

there are an open $U \ni 0$ and $\Phi\colon U \to \mathrm{GL}(E)$ satisfying (5.1) and $f_j(0) = 0$. Hence

$$\Phi(z)g(z) = \Phi(z)f(z)(e) = ez_1^d + \sum_{j=0}^{d-1} f_j(z')(e)z_1^j,$$

as claimed. □

Proof of Lemma 5.1. We will only prove (a), part (b) is proved similarly. The proof will be by induction on n, the case $n = 0$ being trivial.

So assume the $(n-1)$-dimensional case and consider $\Omega \subset \mathbb{C}^n$. Without loss of generality we take $\zeta = 0$. Suppose first $\dim P = 1$, say, $P = \mathbb{C}$, and let $g(z) = f(z)(1)$. Thus $g\colon \Omega \to Q$ is holomorphic. When $g \equiv 0$ near 0, the claim is obvious; otherwise we can choose coordinates so that $\partial^j g/\partial z_1^j(0) \neq 0$ for some j. By Lemma 5.3 there is a holomorphic $\Phi\colon U \to \mathrm{GL}(Q)$ satisfying (5.3). We can assume $U =$

$U_1 \times \Omega' \subset \mathbb{C} \times \mathbb{C}^{n-1}$. Consider the holomorphic function $f' \colon \Omega' \to \mathrm{Hom}(\mathbb{C}^{d+1}, Q)$ given by

$$f'(z')(\xi_0, \xi_1, \dots, \xi_d) = e\xi_0 + \sum_1^d g_j(z')\xi_j.$$

By the inductive assumption, after shrinking U and Ω', there are a $q' \colon U' \to \mathrm{GL}(Q)$ and a finite-dimensional $Q' \subset Q$ so that $\mathrm{Im}\, q'(z')f'(z') \subset Q'$ for all $z' \in U'$. This implies $q'(z')\Phi(z)g(z) \in Q'$, and so with $q(z) = q'(z')\Phi(z)$ indeed $\mathrm{Im}\, q(z)f(z) \subset Q'$.

To prove the claim for $\dim P > 1$ we use induction once more, this time on $\dim P$. Assume the claim holds when $\dim P < k$, and consider a k-dimensional P, $k \geq 2$. Decompose $P = P_1 \oplus P_2$ with $\dim P_1 = 1$. By what we have already proved, there are an open $U \ni 0$, a holomorphic $q_1 \colon U \to \mathrm{GL}(Q)$, and a finite-dimensional $Q_1 \subset Q$ such that $q_1(z)f(z)P_1 \subset Q_1$. Choose a closed complement $Q_2 \subset Q$ to Q_1, and with the projection $\pi \colon Q_1 \oplus Q_2 \to Q_2$ let

$$f_2(z) = \pi q_1(z) f(z) \in \mathrm{Hom}(P, Q_2). \tag{5.4}$$

As $\dim P_2 = k - 1$, by the inductive hypothesis there are a finite-dimensional $Q_2' \subset Q_2$ and (after shrinking U) a holomorphic $q_2' \colon U \to \mathrm{GL}(Q_2)$ such that $q_2'(z)f_2(z)P_2 \subset Q_2'$. We extend q_2' to $q_2 \colon U \to \mathrm{GL}(Q)$ by taking it to be the identity on Q_1. Then $q_2(z)f_2(z)P_2 \subset Q_2'$ and

$$q_2(z)q_1(z)f(z)P_1 \subset Q_1. \tag{5.5}$$

Further, (5.4) implies $(q_1(z)f(z) - f_2(z))P \subset Q_1$ and so

$$q_2(z)q_1(z)f(z)P_2 \subset q_2(z)Q_1 + q_2(z)f_2(z)P_2 \subset Q_1 \oplus Q_2'. \tag{5.6}$$

(5.5) and (5.6) show that $q = q_2 q_1$ and $Q' = Q_1 \oplus Q_2'$ satisfy the requirements, and the proof is complete. □

Proof of Theorem 4.1. Let $\mathcal{F} = \mathcal{O}^F$ and let $\mathcal{A}$ be generated by holomorphic $f_1, \dots, f_k \colon \Omega \to F$. These functions define a holomorphic $f \colon \Omega \to \mathrm{Hom}(\mathbb{C}^k, F)$ by

$$f(z)(\xi_1, \dots, \xi_k) = \sum_j \xi_j f_j(z).$$

Choose a finite-dimensional $Q' \subset F$, an open $U \ni \zeta$, and a holomorphic $q \colon U \to \mathrm{GL}(F)$ as in Lemma 5.1(a). Then $\mathcal{E} = q^{-1}\mathcal{O}^{Q'}|U \subset \mathcal{F}$ is finitely generated and free; moreover, it contains the germs of each $f_j|U$, hence also $\mathcal{A}|U$.

As to flatness: it is known, and easy, that the direct limit of flat modules is flat ([Mt, Appendix B]). As each stalk of $\mathcal{F}$ is the direct limit of its finitely generated free submodules, it is flat. □

Here is another consequence of Lemma 5.1.

Theorem 5.4. *Let $\mathcal{A}$ be a coherent sheaf and let $\mathcal{B} \subset \mathcal{A}$ be a submodule. If there are a plain sheaf $\mathcal{O}^F = \mathcal{F}$ and an $\mathcal{O}$-epimorphism $\varphi \colon \mathcal{F} \to \mathcal{B}$, then $\mathcal{B}$ is coherent. In particular, cohesive subsheaves of $\mathcal{A}$ are coherent.*

If $\mathcal{F}$ of the theorem is finitely generated, then so is $\mathcal{B}$, and its coherence is immediate from the definitions. For the proof of the general statement we need the notion of depth. Recall that given an $\mathcal{O}$-module $\mathcal{A}$, the depth of a stalk $\mathcal{A}_\zeta$ is 0 if there is a submodule $0 \neq L \subset \mathcal{A}_\zeta$ annihilated by the maximal ideal $\mathfrak{m}_\zeta \subset \mathcal{O}_\zeta$. Otherwise depth $\mathcal{A}_\zeta > 0$. (For the general notion of depth, see [Mt, p. 130]; the version we use here is the one, e.g., in [Ms, Proposition 4.2], at least in the positive dimensional case.)

Lemma 5.5. *If $\mathcal{A}$ is a coherent sheaf, then*

$$D = \{z \in \Omega\colon \ \textit{depth } \mathcal{A}_z = 0\}$$

is a discrete set.

Proof. Observe that, given a compact polydisc $K \subset \Omega$, the $\mathcal{O}(K)$-module $\Gamma(K, \mathcal{A})$ is finitely generated. Indeed, if $0 \to \mathcal{A}'|K \to \mathcal{O}^{\oplus p}|K \to \mathcal{A}|K \to 0$ is an exact sequence of $\mathcal{O}|K$-modules, then $H^1(K, \mathcal{A}') = 0$ implies that

$$\mathcal{O}(K)^{\oplus p} \approx \Gamma(K, \mathcal{O}^{\oplus p}) \to \Gamma(K, \mathcal{A})$$

is surjective. We shall also need the fact that $\mathcal{O}(K)$ is Noetherian, see, e.g., [F].

As for the lemma, we can assume $\dim \Omega > 0$. If $z \in D$, there is a nonzero submodule $B \subset \mathcal{A}_z$ such that $\mathfrak{m}_z B = 0$. Let $\mathcal{B}^z$ denote the skyscraper sheaf over Ω whose only nonzero stalk is B, at z. We do this construction for every $z \in D$. With $K \subset \Omega$ a compact polydisc, the submodule

$$\sum_{z \in D \cap K} \Gamma(K, \mathcal{B}^z) \subset \Gamma(K, \mathcal{A}) \tag{5.7}$$

is finitely generated. But $\Gamma(K, \mathcal{B}^z) \neq 0$ consists of (certain) sections of $\mathcal{A}$ supported at z. It follows that the sum in (5.7) is a direct sum, hence in fact a finite direct sum. In other words, $D \cap K$ is finite for every compact polydisc K, and D must be discrete. □

Proof of Theorem 5.4. We can suppose $\dim \Omega > 0$. First we assume that, in addition, depth $\mathcal{A}_z > 0$ for every z. Since coherence is a local property, and $\mathcal{A}$ is locally finitely generated, we can assume that Ω is a ball, and there are a finitely generated plain sheaf $\mathcal{O}^E = \mathcal{E} \approx \mathcal{O} \oplus \cdots \oplus \mathcal{O}$ and an epimorphism $\epsilon\colon \mathcal{E} \to \mathcal{A}$. We are precisely in the situation of Theorem 7.1 in [Ms]. By this theorem, φ factors through ϵ: there is an $\mathcal{O}$-homomorphism $\psi\colon \mathcal{F} \to \mathcal{E}$ such that $\varphi = \epsilon\psi$. (Masagutov in his proof of Theorem 7.1 relies on a result of the present paper, but the reasoning is not circular. What the proof of [Ms, Theorem 7.1] needs is our Theorem 4.3, whose proof is independent of Theorem 5.4 we are justifying here.) As an $\mathcal{O}$-homomorphism between plain sheaves, ψ is plain by [Ms, Theorem 1.1].

In view of Lemma 5.1(b), there are a finite codimensional $F' \subset F$ and a plain isomorphism $\rho\colon \mathcal{F} \to \mathcal{F}$ such that $\psi\rho|\mathcal{O}^{F'} = 0$. If $F'' \subset F$ denotes a (finite-dimensional) complement to F', then $\psi\rho(\mathcal{O}^{F''}) = \psi\rho(\mathcal{F}) = \psi(\mathcal{F})$. Hence $\epsilon\psi\rho(\mathcal{O}^{F''}) = \epsilon\psi(\mathcal{F}) = \mathcal{B}$ is finitely generated; as a submodule of a coherent sheaf, itself must be coherent.

Now take an $\mathcal{A}$ whose depth is 0 at some z. In view of Lemma 5.5 we can assume that there is a single such z. With

$$C = \{a \in \mathcal{A}_z : \mathfrak{m}_z^k a = 0 \text{ for some } k = 1, 2, \ldots\},$$

let $\mathcal{C} \subset \mathcal{A}$ be the skyscraper sheaf over Ω whose only nonzero stalk is C, at z. As C is finitely generated, $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are coherent. Also, $\operatorname{depth}(\mathcal{A}/\mathcal{C})_\zeta > 0$ for every $\zeta \in \Omega$. Therefore by the first part of the proof $\mathcal{B}/\mathcal{B} \cap \mathcal{C} \subset \mathcal{A}/\mathcal{C}$ is coherent. Since $\mathcal{B} \cap \mathcal{C}$, supported at the single point z, is coherent, the Three Lemma implies $\mathcal{B}$ is coherent, as claimed. □

6. Hom and ⊗

The main result of this section is the following. Let $\mathcal{A}$ be an $\mathcal{O}$-module and $\mathcal{B}$ an analytic sheaf. Recall that, given a plain sheaf $\mathcal{E}$, in Section 3 we introduced a tautological $\mathcal{O}$-homomorphism

$$T = T_{\mathcal{E}} : \mathcal{A} \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B}) \to \mathbf{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{A} \otimes \mathcal{B}), \tag{6.1}$$

and $\mathbf{Hom}(\mathcal{E}, \mathcal{A} \otimes \mathcal{B})$ was defined as the image of $T_{\mathcal{E}}$.

Theorem 6.1. *If $\mathcal{B}$ is plain, then T is injective.*

Suppose $\mathcal{E} = \mathcal{O}^E, \mathcal{B} = \mathcal{O}^B$ are plain sheaves. If $\zeta \in \Omega$, a $\mathbb{C}$-linear map

$$S^\zeta : \mathcal{A}_\zeta \otimes \mathcal{O}_\zeta^{\mathrm{Hom}(E,B)} \to \mathrm{Hom}_{\mathbb{C}}(E, \mathcal{A}_\zeta \otimes \mathcal{O}_\zeta^B) \tag{6.2}$$

can be defined as follows. Let $a \in \mathcal{A}_\zeta$, $\Theta \in \mathcal{O}_\zeta^{\mathrm{Hom}(E,B)}$, then

$$S^\zeta(a \otimes \Theta)(e) = a \otimes \Theta e, \qquad e \in E,$$

where on the right e is thought of as a constant germ $\in \mathcal{O}_\zeta^E$. The key to Theorem 6.1 is the following

Lemma 6.2. *Let E, B be Banach spaces, $\zeta \in \Omega$, and let M be an $\mathcal{O}_\zeta$-module. Then the tautological homomorphism*

$$S : M \otimes \mathcal{O}_\zeta^{\mathrm{Hom}(E,B)} \to \mathrm{Hom}_{\mathbb{C}}(E, M \otimes \mathcal{O}_\zeta^B) \tag{6.3}$$

given by $S(m \otimes \Theta)(e) = m \otimes \Theta e$, for $e \in E$, is injective.

As T was, S is also natural with respect to $\mathcal{O}_\zeta$-homomorphisms $M \to N$. The claim of the lemma is obvious when M is free, for then tensor products $M \otimes L$ are just direct sums of copies of L. The claim is also obvious when M is a direct summand in a free module $M' = M \oplus N$, as the tautological homomorphism for M' decomposes into the direct sum of the tautological homomorphisms for M and N.

Proof of Lemma 6.2. The proof is inspired by the proof of [Ms, Theorem 1.3]. The heart of the matter is to prove when M is finitely generated. Let us write (L_n) for the statement of the lemma for M finitely generated and $n = \dim \Omega$; we prove it by induction on n. (L_0) is trivial, as $\mathcal{O}_\zeta \approx \mathbb{C}$ is a field and any module over it is free. So assume (L_{n-1}) for some $n \geq 1$, and prove (L_n). We take $\zeta = 0$.

Step 1°. First we verify (L_n) with the additional assumption that $gM = 0$ with some $0 \neq g \in \mathcal{O}_0$. By Weierstrass' preparation theorem we can take g to be (the germ of) a Weierstrass polynomial of degree $d \geq 1$ in the z_1 variable. We write $z = (z_1, z') \in \mathbb{C}^n$, and $\mathcal{O}'_0, \mathcal{O}'^F_0$ for the ring/module of the corresponding germs in $\mathbb{C}^{n-1}$ (here F is any Banach space). We embed $\mathcal{O}'_0 \subset \mathcal{O}_0$, $\mathcal{O}'^F_0 \subset \mathcal{O}^F_0$ as germs independent of z_1. This makes $\mathcal{O}_0$-modules into $\mathcal{O}'_0$-modules. In the proof tensor products both over $\mathcal{O}_0$ and $\mathcal{O}'_0$ will occur; we keep writing $\otimes$ for the former and will write $\otimes'$ for the latter.

We claim that the $\mathcal{O}'_0$-homomorphism

$$i : M \otimes' \mathcal{O}'^F_0 \to M \otimes \mathcal{O}^F_0, \qquad i(m \otimes' f') = m \otimes f',$$

is in fact an isomorphism. To verify it is surjective, consider $m \otimes f \in M \otimes \mathcal{O}^F_0$. By Weierstrass' division theorem, valid for vector-valued functions as well (e.g., the proof in [GuR, p. 70] carries over verbatim), f can be written

$$f = f_0 g + \sum_{j=0}^{d-1} f'_j z_1^j, \qquad f_0 \in \mathcal{O}^F_0, f'_j \in \mathcal{O}'^F_0.$$

Thus $m \otimes f = m \otimes (f_0 g + \sum f'_j z_1^j) = i(\sum z_1^j m \otimes' f'_j)$ is indeed in Im i. Further, injectivity is clear if dim $F = k < \infty$, as $M \otimes' \mathcal{O}'^F_0 \approx M^{\oplus k}$, $M \otimes \mathcal{O}^F_0 \approx M^{\oplus k}$, and i corresponds to the identity of $M^{\oplus k}$. For a general F consider a finitely generated submodule $A \subset \mathcal{O}'^F_0$. Lemma 5.1(a) implies that there are a neighborhood U of $0 \in \mathbb{C}^{n-1}$, a finite-dimensional subspace $G \subset F$, and a holomorphic $q : U \to \mathrm{GL}(F)$ such that the automorphism φ' of $\mathcal{O}'^F_0$ induced by q maps A into $\mathcal{O}'^G_0 \subset \mathcal{O}'^F_0$. (The reasoning is the same as in the proof of Theorem 4.1.) If extended to $\mathbb{C} \times U$ independent of z_1, q also induces an automorphism φ of $\mathcal{O}^F_0$, and i intertwines the automorphisms $\mathrm{id}_M \otimes' \varphi'$ and $\mathrm{id}_M \otimes \varphi$. Now i is injective between $M \otimes' \mathcal{O}'^G_0$ and $M \otimes \mathcal{O}^G_0 \subset M \otimes \mathcal{O}^F_0$, because dim $G < \infty$. As the image of $M \otimes' A$ in $M \otimes' \mathcal{O}'^F_0$ is contained in $M \otimes' \mathcal{O}'^G_0$, it follows that i is injective on this image. Since the finitely generated $A \subset \mathcal{O}'^F_0$ was arbitrary, i is indeed injective.

Applying this with $F = \mathrm{Hom}(E, B)$ and $F = B$, we obtain a commutative diagram

$$\begin{array}{ccc} M \otimes' \mathcal{O}'^{\mathrm{Hom}(E,B)}_0 & \xrightarrow{\approx} & M \otimes \mathcal{O}^{\mathrm{Hom}(E,B)}_0 \\ \Big\downarrow{S'} & & \Big\downarrow{S} \\ \mathrm{Hom}_{\mathbb{C}}(E, M \otimes' \mathcal{O}'^B_0) & \xrightarrow{\approx} & \mathrm{Hom}_{\mathbb{C}}(E, M \otimes \mathcal{O}^B_0). \end{array}$$

Here S' is also a tautological homomorphism. Now M is finitely generated over $\mathcal{O}_0/g\mathcal{O}_0$, and this latter is a finitely generated $\mathcal{O}'_0$-algebra by Weierstrass division. It follows that M is finitely generated over $\mathcal{O}'_0$; by the induction hypothesis S' is injective, hence so must be S.

Step 2°. Now take an arbitrary finitely generated M. Let $\mu\colon L \to M$ be an epimorphism from a free finitely generated $\mathcal{O}_0$-module L, and $K = \text{Ker}\ \mu$. If the exact sequence

$$0 \to K \xrightarrow{\lambda} L \xrightarrow{\mu} M \to 0 \tag{6.4}$$

splits, then M is a direct summand in L and, as said, the claim is immediate. The point of the reasoning to follow is that, even if (6.4) does not split, it does split up to torsion in the following sense: there are $\sigma \in \text{Hom}(L, K)$ and $0 \neq g \in \mathcal{O}_0$ such that $\sigma\lambda\colon K \to K$ is multiplication by g. To see this, let Q be the field of fractions of $\mathcal{O}_0$, and note that the induced linear map $\lambda_Q\colon K \otimes Q \to L \otimes Q$ of Q-vector spaces has a left inverse τ. Clearing denominators in τ then yields the σ needed.

We denote the tautological homomorphisms (6.3) for K, L, M by S_K, S_L, S_M. Tensoring and Hom-ing (6.4) gives rise to a commutative diagram

$$\begin{array}{ccccccc}
K \otimes \mathcal{O}_0^{\text{Hom}(E,B)} & \underset{\sigma_t}{\overset{\lambda_t}{\rightleftarrows}} & L \otimes \mathcal{O}_0^{\text{Hom}(E,B)} & \xrightarrow{\mu_t} & M \otimes \mathcal{O}_0^{\text{Hom}(E,B)} & \longrightarrow & 0 \\
\downarrow{\scriptstyle S_K} & & \downarrow{\scriptstyle S_L} & & \downarrow{\scriptstyle S_M} & & \\
\text{Hom}_{\mathbb{C}}(E, K \otimes \mathcal{O}_0^B) & \underset{\sigma_h}{\overset{\lambda_h}{\rightleftarrows}} & \text{Hom}_{\mathbb{C}}(E, L \otimes \mathcal{O}_0^B) & \xrightarrow{\mu_h} & \text{Hom}_{\mathbb{C}}(E, M \otimes \mathcal{O}_0^B) & \longrightarrow & 0
\end{array}$$

with exact rows. Here λ_t, λ_h, etc. just indicate homomorphisms induced on various modules by λ, etc. Consider an element of Ker S_M; it is of form $\mu_t u$, $u \in L \otimes \mathcal{O}_0^{\text{Hom}(E,B)}$. Then $S_L u \in \text{Ker}\ \mu_h = \text{Im}\ \lambda_h$. Let $S_L u = \lambda_h v$. We compute

$$S_L \lambda_t \sigma_t u = \lambda_h S_K \sigma_t u = \lambda_h \sigma_h S_L u = \lambda_h \sigma_h \lambda_h v = \lambda_h g v = S_L g u.$$

Since L is free, S_L is injective, so $\lambda_t \sigma_t u = gu$ and $g\mu_t u = \mu_t \lambda_t \sigma_t u = 0$. We conclude that $g\text{Ker}\ S_M = 0$. Let $N \subset M$ denote the submodule of elements annihilated by g and, for brevity, set $H = \mathcal{O}_0^{\text{Hom}(E,B)}$, a flat module. Multiplication by g is a monomorphism on M/N, so the same holds on $M/N \otimes H$. The exact sequence

$$0 \to N \otimes H \hookrightarrow M \otimes H \to M/N \otimes H \to 0$$

then shows that in $M \otimes H$ the kernel of multiplication by g is $N \otimes H$. Therefore $N \otimes H \supset \text{Ker}\ S_M$, and $\text{Ker}\ S_M \subset \text{Ker}\ S_N$. But $gN = 0$, so from Step 1° it follows that $\text{Ker}\ S_N = 0$, and again $\text{Ker}\ S_M = 0$.

Step 3°. Having proved the lemma for finitely generated modules, consider an arbitrary $\mathcal{O}_0$-module M. The inclusion $\iota : N \hookrightarrow M$ of a finitely generated submodule induces a commutative diagram

$$\begin{array}{ccc} N\otimes\mathcal{O}_0^{\mathrm{Hom}(E,B)} & \xrightarrow{\iota_t} & M\otimes\mathcal{O}_0^{\mathrm{Hom}(E,B)} \\ \Big\downarrow{\scriptstyle S_N} & & \Big\downarrow{\scriptstyle S} \\ \mathrm{Hom}_{\mathbb{C}}(E, N\otimes\mathcal{O}_0^B) & \xrightarrow{\iota_h} & \mathrm{Hom}_{\mathbb{C}}(E, M\otimes\mathcal{O}_0^B), \end{array}$$

with S_N the tautological homomorphism for N. Flatness implies that ι_t, ι_h are injective; as S_N is also injective by what we have proved so far, S itself is injective on the range of ι_t. As N varies, these ranges cover all of $M\otimes\mathcal{O}_0^{\mathrm{Hom}(E,B)}$, hence S is indeed injective. □

Proof of Theorem 6.1. For $\zeta\in\Omega$ we embed $E\to\mathcal{O}_\zeta^E$ as constant germs; this induces a $\mathbb{C}$-linear map

$$\rho:\mathbf{Hom}_{\mathcal{O}}(\mathcal{O}^E,\mathcal{A}\otimes\mathcal{O}^B)_\zeta\to\mathrm{Hom}_{\mathbb{C}}(E,\mathcal{A}_\zeta\otimes\mathcal{O}_\zeta^B).$$

It will suffice to show that if we restrict T to the stalk at ζ and compose it with ρ, the resulting map

$$T^\zeta:\mathcal{A}_\zeta\otimes\mathbf{Hom}(\mathcal{O}^E,\mathcal{O}^B)_\zeta\to\mathrm{Hom}_{\mathbb{C}}(E,\mathcal{A}_\zeta\otimes\mathcal{O}_\zeta^B),$$

given by $T^\zeta(a\otimes\theta)(e)=a\otimes\theta e$, is injective. But, by the canonical isomorphism $\mathcal{O}_\zeta^{\mathrm{Hom}(E,B)}\overset{\approx}{\to}\mathbf{Hom}(\mathcal{O}^E,\mathcal{O}^B)_\zeta$, cf. (2.2), T^ζ is injective precisely when S^ζ of (6.2) is; so that Lemma 6.2 finishes off the proof. □

Now we can return to the question how compatible are inducing in the sense of Definition 2.4 and tensoring.

Lemma 6.3. *If* $0\to\mathcal{A}'\overset{\varphi}{\to}\mathcal{A}\overset{\psi}{\to}\mathcal{A}''\to 0$ *is an exact sequence of* $\mathcal{O}$*-modules and* $\mathcal{B}$ *is a plain sheaf, then* $\varphi\otimes id_{\mathcal{B}}$*, resp.* $\psi\otimes id_{\mathcal{B}}$*, induce from* $\mathcal{A}\otimes\mathcal{B}$ *the tensor product analytic structure on* $\mathcal{A}'\otimes\mathcal{B}$*, resp.* $\mathcal{A}''\otimes\mathcal{B}$*.*

Proof. The case of $\mathcal{A}''\otimes\mathcal{B}$, in greater generality, is the content of Proposition 3.6. Consider $\mathcal{A}'\otimes\mathcal{B}$. Meaning by $\mathcal{A}'\otimes\mathcal{B}$ etc. the analytic sheaves endowed with the tensor product structure, in light of Definition 2.4 we are to prove

$$\mathbf{Hom}(\mathcal{E},\mathcal{A}'\otimes\mathcal{B})=(\varphi\otimes\mathrm{id}_{\mathcal{B}})_*^{-1}\mathbf{Hom}(\mathcal{E},\mathcal{A}\otimes\mathcal{B}) \tag{6.5}$$

for every plain $\mathcal{E}$. Again using that $\mathcal{B}$ and $\mathbf{Hom}(\mathcal{E},\mathcal{B})$ are flat, from $0\to\mathcal{A}'\to\mathcal{A}\to\mathcal{A}''\to 0$ we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0\longrightarrow & \mathcal{A}'\otimes\mathbf{Hom}(\mathcal{E},\mathcal{B}) & \xrightarrow{\varphi_t} & \mathcal{A}\otimes\mathbf{Hom}(\mathcal{E},\mathcal{B}) & \xrightarrow{\psi_t} & \mathcal{A}''\otimes\mathbf{Hom}(\mathcal{E},\mathcal{B}) \\ & \Big\downarrow{\scriptstyle T'} & & \Big\downarrow{\scriptstyle T} & & \Big\downarrow{\scriptstyle T''} \\ 0\longrightarrow & \mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{A}'\otimes\mathcal{B}) & \xrightarrow{\varphi_h} & \mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{A}\otimes\mathcal{B}) & \xrightarrow{\psi_h} & \mathbf{Hom}_{\mathcal{O}}(\mathcal{E},\mathcal{A}''\otimes\mathcal{B}). \end{array}$$

The vertical arrows are the respective tautological homomorphisms, and $\varphi_t=\varphi\otimes\mathrm{id}_{\mathbf{Hom}(\mathcal{E},\mathcal{B})}$, $\varphi_h=(\varphi\otimes\mathrm{id}_{\mathcal{B}})_*$, etc. denote maps induced by φ, etc. From this diagram, the left-hand side of (6.5), Im T', is clearly contained in φ_h^{-1}Im T, i.e., in the right-hand side. To show the converse, suppose $\epsilon \in \mathbf{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{A}' \otimes \mathcal{B})$ is in φ_h^{-1}Im T, say, $\varphi_h \epsilon = Tu$ with $u \in \mathcal{A} \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B})$. Then $T''\psi_t u = \psi_h T u = \psi_h \varphi_h \epsilon = 0$. Since T'' is injective by Theorem 6.1, $\psi_t u = 0$. It follows that $u = \varphi_t v$ with some $v \in \mathcal{A}' \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B})$, whence $\varphi_h T' v = T\varphi_t v = Tu = \varphi_h \epsilon$. As φ_h is also injective, $\epsilon = T'v$; that is, φ_h^{-1}Im $T \subset$ Im T', as needed. □

7. Coherence and cohesion

Proof of Theorems 4.3 *and* 4.4. We have to show that if $\mathcal{A}$ is a coherent sheaf and $\mathcal{B} = \mathcal{O}^B$ plain then $\mathcal{A} \otimes \mathcal{B}$ is cohesive. This would imply that $\mathcal{A} \otimes \mathcal{O}$ is cohesive, and in view of Proposition 3.4 that $\mathcal{A} \approx \mathcal{A} \otimes \mathcal{O}$, with its minimal analytic structure, is also cohesive.

We can cover Ω with open sets over each of which $\mathcal{A}$ has a resolution by finitely generated free $\mathcal{O}$-modules. We can assume that such a resolution

$$\cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{A} \to 0$$

exists over all of Ω, and $\mathcal{F}_j = \mathcal{O}^{F_j}$, dim $F_j < \infty$. If $\mathcal{E} = \mathcal{O}^E$ is plain then

$$\cdots \to \mathcal{F}_2 \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B}) \to \mathcal{F}_1 \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B}) \to \mathcal{A} \otimes \mathbf{Hom}(\mathcal{E}, \mathcal{B}) \to 0$$

is also exact, $\mathbf{Hom}(\mathcal{E}, \mathcal{B}) \approx \mathcal{O}^{\mathrm{Hom}(E,B)}$ being flat. By Theorem 6.1 this sequence is isomorphic to

$$\cdots \to \mathbf{Hom}(\mathcal{E}, \mathcal{F}_2 \otimes \mathcal{B}) \to \mathbf{Hom}(\mathcal{E}, \mathcal{F}_1 \otimes \mathcal{B}) \to \mathbf{Hom}(\mathcal{E}, \mathcal{A} \otimes \mathcal{B}) \to 0,$$

which then must be exact. Here $\mathcal{F}_j \otimes \mathcal{B} \approx \mathcal{O}^{F_j \otimes B}$ analytically, cf. (3.2). Now Theorem 2.7 applies. We conclude that

$$\cdots \to \mathcal{F}_2 \otimes \mathcal{B} \to \mathcal{F}_1 \otimes \mathcal{B} \to \mathcal{A} \otimes \mathcal{B} \to 0$$

is completely exact, and $\mathcal{A} \otimes \mathcal{B}$ is indeed cohesive. □

8. Application. Complex analytic subspaces and subvarieties

The terminology in the subject indicated in the title is varied and occasionally ambiguous, even in finite-dimensional complex geometry. Here we will use the terms "complex subspace" and "subvariety" to mean different things. Following [GrR], a complex subspace A of an open $\Omega \subset \mathbb{C}^n$ is obtained from a coherent submodule $\mathcal{J} \subset \mathcal{O}$. The support $|A|$ of the sheaf $\mathcal{O}/\mathcal{J}$, endowed with the sheaf of rings $(\mathcal{O}/\mathcal{J})\big||A| = \mathcal{O}_A$ defines a ringed space, and the pair $(|A|, \mathcal{O}_A)$ is the complex subspace in question.

For infinite-dimensional purposes this notion is definitely not adequate, and in the setting of Banach spaces in [LP] we introduced a new notion that we called subvariety. Instead of coherent sheaves, they are defined in terms of cohesive sheaves, furthermore, one has to specify a subsheaf $\mathcal{J}^E \subset \mathcal{O}^E$ for each Banach space E, (thought of as germs vanishing on the subvariety), not just one $\mathcal{J} \subset \mathcal{O}$. The reason this definition was made was to delineate a class of subsets in Banach spaces that arise in complex analytical questions, and can be studied using complex analysis. At the same time, the definition makes sense in $\mathbb{C}^n$ as well, and it is natural to ask how subvarieties and complex subspaces in $\mathbb{C}^n$ are related. Before answering we have to go over the definition of subvarieties, following [LP].

An ideal system over $\Omega \subset \mathbb{C}^n$ is the specification, for every Banach space E, of a submodule $\mathcal{J}^E \subset \mathcal{O}^E$, subject to the following: given $z \in \Omega$, $\varphi \in \mathcal{O}_z^{\mathrm{Hom}(E,F)}$, and $e \in \mathcal{J}_z^E$, we have $\varphi e \in \mathcal{J}_z^F$.

Within an ideal system the support of $\mathcal{O}^E/\mathcal{J}^E$ is the same for every $E \neq (0)$, and we call this set the support of the ideal system.

A subvariety S of Ω is given by an ideal system of cohesive subsheaves $\mathcal{J}^E \subset \mathcal{O}^E$. The support of the ideal system is called the support $|S|$ of the subvariety, and we endow it with the sheaves $\mathcal{O}_S^E = \mathcal{O}^E/\mathcal{J}^E \big| |S|$ of modules over $\mathcal{O}_S = \mathcal{O}_S^{\mathbb{C}}$. The "functored space" $(|S|,\ E \mapsto \mathcal{O}_S^E)$ is the subvariety S in question.

Theorem 8.1. *There is a canonical way to associate a subvariety with a complex subspace of* Ω *and vice versa.*

Proof, or rather construction. Let $i\colon \mathcal{J} \hookrightarrow \mathcal{O}$ be the inclusion of a coherent sheaf $\mathcal{J}$ that defines a complex subspace. The ideal system $\mathcal{J}^E = \mathcal{J}\mathcal{O}^E \subset \mathcal{O}^E$ then gives rise to a subvariety, provided $\mathcal{J}^E$ with the analytic structure inherited from $\mathcal{O}^E$ is cohesive. Consider the diagram

$$\begin{array}{ccc} \mathcal{J} \otimes \mathcal{O}^E & \xrightarrow{\;i\otimes \mathrm{id}_{\mathcal{O}^E}\;} & \mathcal{O} \otimes \mathcal{O}^E \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \approx} \\ \mathcal{J}^E & \xrightarrow{\qquad\qquad} & \mathcal{O}^E. \end{array}$$

Here the vertical arrow on the right, given by $1 \otimes e \mapsto e$, is an analytic isomorphism by Proposition 3.4. The vertical arrow μ on the left is determined by the commutativity of the diagram; it is surjective. As $\mathcal{O}^E$ is flat, $i \otimes \mathrm{id}_{\mathcal{O}^E}$ is injective, therefore μ is an isomorphism. If $\mathcal{J} \otimes \mathcal{O}^E$ is endowed with the analytic structure induced by $i \otimes \mathrm{id}_{\mathcal{O}^E}$, μ becomes an analytic isomorphism. On the other hand, this induced structure of $\mathcal{J} \otimes \mathcal{O}^E$ agrees with the tensor product analytic structure by Lemma 6.3 (set $\mathcal{A}' = \mathcal{J}, \mathcal{A} = \mathcal{O}$), hence it is cohesive by Theorem 4.4. The upshot is that $\mathcal{J}^E$ is indeed cohesive.

As to the converse, suppose $\mathcal{J}^E$ is a cohesive ideal system defining a subvariety. Then $\mathcal{J} = \mathcal{J}^{\mathbb{C}} \subset \mathcal{O}$ is coherent by Theorem 5.4, and gives rise to a complex subspace. □

Theorem 8.1 is clearly not the last word on the matter. First, it should be decided whether the construction in the theorem is a bijection between subvarieties and complex subspaces; second, the functoriality properties of the construction should be investigated.

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Characteristic Classes of the Boundary of a Complex b-manifold

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Dedicated to Linda P. Rothschild

Abstract. We prove a classification theorem by cohomology classes for compact Riemannian manifolds with a one-parameter group of isometries without fixed points generalizing the classification of line bundles (more precisely, their circle bundles) over compact manifolds by their first Chern class. We also prove a classification theorem generalizing that of holomorphic line bundles over compact complex manifold by the Picard group of the base for a subfamily of manifolds with additional structure resembling that of circle bundles of such holomorphic line bundles.

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1. Introduction

This work presents classification theorems for manifolds with additional structure generalizing classical theorems concerning circle bundles of complex line bundles over compact manifolds, both in the C^∞ and holomorphic categories.

In the next paragraphs we will briefly discuss the classical situations from the perspective of this paper. Section 2 gives a context and serves as motivation for the work. Sections 3 and 4 concern the classification theorems themselves. Finally, Section 5 is devoted to an analysis of some aspects of the conditions in Section 4 that generalize the notion of holomorphic function.

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It is well known that the isomorphism classes of smooth complex line bundles over, say, a smooth compact manifold $\mathcal{B}$ are, in a natural way, in one to one correspondence with the elements of the first cohomology group of $\mathcal{B}$ with coefficients in the sheaf $\mathscr{E}^*$ of germs of smooth nonvanishing complex-valued functions on $\mathcal{B}$, and through the first Chern class, with the elements of the second cohomology group of $\mathcal{B}$ with integral coefficients. In somewhat more generality, the construction of the correspondence goes as follows. Fix a line bundle $\pi_E : E \to \mathcal{B}$. If $\{V_a\}_{a\in A}$ is a sufficiently fine open cover of $\mathcal{B}$, then for every line bundle $\pi_{E'} : E' \to \mathcal{B}$ there is a family of isomorphisms $h_a : E'_{V_a} \to E_{V_a}$ (where for instance $E_{V_a} = \pi_E^{-1}(V_a)$) such that $\pi_E \circ h_a = \pi_{E'}$. The maps $h_{ab} = h_a \circ h_b^{-1}$ are isomorphisms of $E_{V_a \cap V_b}$ onto itself such that $\pi_E \circ h_{ab} = \pi_E$, therefore given by multiplication by nonvanishing functions f_{ab}. These functions define a Čech 1-cocycle giving an element of the first cohomology group of $\mathcal{B}$ with coefficients in $\mathscr{E}^*$. This correspondence is a bijection which in the case where E is the trivial line bundle is the one alluded to above.

Once the correspondence with the elements of the first cohomology group is established, the first Chern class map arises through composition with the connecting homomorphism $H^1(\mathcal{B}, \mathscr{E}^*) \to H^2(\mathcal{B}, \mathbb{Z})$ in the long exact sequence in cohomology associated with the short exact sequence

$$0 \to \mathscr{Z} \to \mathscr{E} \xrightarrow{\exp 2\pi i} \mathscr{E}^* \to 0, \tag{1.1}$$

in which $\mathscr{Z}$ is the sheaf of germs of locally constant $\mathbb{Z}$-valued functions and $\mathscr{E}$ is the sheaf of germs of smooth functions on $\mathcal{B}$. The connecting map is an isomorphism because $\mathscr{E}$ is a fine sheaf. Under this map, the class associated to the line bundle E' as described in the previous paragraph goes to $\mathrm{c}_1(E') - \mathrm{c}_1(E)$, the difference of the standard first Chern classes of E' and E.

The version of the above classification generalized in Section 3 goes as follows. Since the isomorphism classes of line bundles over $\mathcal{B}$ are in one to one correspondence with the isomorphism classes of principal S^1-bundles over $\mathcal{B}$, the above analysis also applies to such bundles. Fix a principal S^1-bundle $\pi_{\mathcal{N}} : \mathcal{N} \to \mathcal{B}$ and consider, for any other principal $\pi_{\mathcal{N}'} : S^1 \to \mathcal{N}'$ and suitable open cover $\{V_a\}_{a\in A}$ of $\mathcal{B}$, a family of equivariant diffeomorphisms $h_a : U'_a \to U_a$ (where $U_a = \pi^{-1}(V_a)$) such that $\pi_{\mathcal{N}} \circ h_a = \pi_{\mathcal{N}'}|_{U'_a}$. Then $h_{ab} = h_a \circ h_b^{-1}|_{U_a}$ is an equivariant diffeomorphism from $U_a \cap U_b$ to itself such that $\pi_{\mathcal{N}} \circ h_{ab} = \pi_{\mathcal{N}}$. From the collection $\{h_{ab}\}$ one can of course construct an isomorphic copy of $\mathcal{N}'$. The family of germs of equivariant diffeomorphisms $h : U \to U$ ($U = \pi_{\mathcal{N}}^{-1}(V)$, $V \subset \mathcal{B}$ open) such that $\pi_{\mathcal{N}} \circ h = \pi_{\mathcal{N}}|_U$ forms an abelian sheaf $\mathscr{I}^\infty(\mathcal{N})$ over $\mathcal{B}$. Elements of $\mathscr{I}^\infty(\mathcal{N})$ are locally represented by multiplication by functions $e^{2\pi i f}$, where f is a smooth real-valued function on an open set of the base. This leads to an exact sequence

$$0 \to \mathscr{Z} \to \mathscr{C}^\infty(\mathcal{B}, \mathbb{R}) \xrightarrow{\exp 2\pi i} \mathscr{I}^\infty(\mathcal{N}) \to 0, \tag{1.2}$$

similar to (1.1), in which $\mathscr{C}^\infty(\mathcal{B}, \mathbb{R})$ is the sheaf of germs of smooth real-valued functions on $\mathcal{B}$. Here $\mathbb{R}$ is appears as the Lie algebra of S^1. Since $\mathscr{C}^\infty(\mathcal{B}, \mathbb{R})$ is fine, again one has an isomorphism $H^1(\mathcal{B}, \mathscr{I}^\infty(\mathcal{N})) \to H^2(\mathcal{B}, \mathbb{Z})$. In Section 3, the S^1-action is replaced by an $\mathbb{R}$-action, the one-parameter group of diffeomorphisms generated by a nonvanishing vector field $\mathcal{T}$ admitting a $\mathcal{T}$-invariant Riemannian metric. The orbits of the group need not be compact. The condition $\pi_{\mathcal{N}} \circ h = \pi_{\mathcal{N}}|_U$ is replaced by the condition that for every $p \in U$, $h(p)$ belongs to the closure of the orbit of p, $\mathbb{R}$ is replaced by the Lie algebra $\mathfrak{g}$ of a certain torus G, and $\mathbb{Z}$ by the kernel of the exponential map $\exp : \mathfrak{g} \to G$.

Suppose now that $\mathcal{B}$ is a compact complex manifold and $\pi : \mathcal{N} \to \mathcal{B}$ is the circle bundle of a holomorphic line bundle $E \to \mathcal{B}$. Implicit here is that $\mathcal{N}$ is the set of unit vectors of E with respect to some Hermitian metric. In addition to the infinitesimal generator $\mathcal{T}$ of the canonical S^1-action on $\mathcal{N}$ one has the subbundle $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ of the complexification of the tangent bundle of $\mathcal{N}$ whose elements are precisely those whose image by π_* lies in $T^{0,1}\mathcal{B}$. This is an involutive subbundle having $\mathcal{T}$ as a section. The unit ball bundle of E has $\mathcal{N}$ as boundary, so $\mathcal{N}$, as a hypersurface in the complex manifold E, is naturally a CR manifold. Its CR structure $\overline{\mathcal{K}}$ (as vectors of type $(0,1)$) is also a subbundle of $\overline{\mathcal{V}}$, and in fact $\overline{\mathcal{V}} = \overline{\mathcal{K}} \oplus \operatorname{span}_{\mathbb{C}} \mathcal{T}$. The unique real one-form θ determined by the condition that it vanishes on $\overline{\mathcal{K}}$ and satisfies $\langle \theta, \mathcal{T} \rangle = 1$ is related to the complex and Hermitian structures of E by the fact that $i\theta$ is the connection form of the Hermitian holomorphic connection of E. The restriction β of $-i\theta$ to $\overline{\mathcal{V}}$ is of course a smooth section of $\overline{\mathcal{V}}^*$. Since $\overline{\mathcal{V}}$ is involutive, there is a differential operator from sections of $\overline{\mathcal{V}}^*$ to sections of $\bigwedge^2 \overline{\mathcal{V}}^*$. This operator, analogous but not equal, to the $\overline{\partial}$ operator (or the $\overline{\partial}_b$ operator), is denoted $\overline{\mathbb{D}}$ and forms part of an elliptic complex. The form β is $\mathbb{D}$-closed: $\overline{\mathbb{D}}\beta = 0$. This condition reflects the fact that θ corresponds to a holomorphic connection, so its curvature has vanishing $(0,2)$ component. Conversely, if $\mathcal{N} \to \mathcal{B}$ is a circle bundle over a complex manifold, if $\overline{\mathcal{V}}$ is defined as above, and if β is a smooth section of $\overline{\mathcal{V}}^*$ such that $\langle \beta, \mathcal{T} \rangle = -i$, then $\mathcal{N}$ is the circle bundle of a holomorphic line bundle and β arises from the Hermitian holomorphic connection as described. The equivariant diffeomorphisms $h_{ab} : U_a \cap U_b \to U_a \cap U_b$ arising from a system of holomorphic transition functions satisfy $dh_{ab}\overline{\mathcal{V}} = \overline{\mathcal{V}}$.

In Section 4 we prove a classification result for general compact manifolds together with a nonvanishing real vector field $\mathcal{T}$ admitting a $\mathcal{T}$-invariant Riemannian metric and an involutive subbundle $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ such that $\mathcal{V} \cap \overline{\mathcal{V}} = \operatorname{span}_{\mathbb{C}} \mathcal{T}$ and smooth sections β of $\overline{\mathcal{V}}^*$ such that $\langle \beta, \mathcal{T} \rangle = -i$ and $\overline{\mathbb{D}}\beta = 0$. Again, the orbits of $\mathcal{T}$ need not be compact. The objects $\mathcal{T}$, $\overline{\mathcal{V}}$, β arise naturally on the boundary of complex b-manifolds. These manifolds, and how the structure on the boundary arises, are described in the next section.

2. Complex *b*-manifolds

Let $\mathcal{M}$ be a smooth manifold with boundary. The b-tangent bundle of $\mathcal{M}$ (Melrose [2, 3]) is a smooth vector bundle ${}^bT\mathcal{M} \to \mathcal{M}$ together with a smooth vector bundle homomorphism

$$\operatorname{ev} : {}^bT\mathcal{M} \to T\mathcal{M}$$

covering the identity such that the induced map

$$\mathrm{ev}_*: C^\infty(\mathcal{M};{}^bT\mathcal{M}) \to C^\infty(\mathcal{M};T\mathcal{M})$$

is a $C^\infty(\mathcal{M};\mathbb{R})$-module isomorphism onto the submodule $C^\infty_{\tan}(\mathcal{M};T\mathcal{M})$ of vector fields on $\mathcal{M}$ which are tangential to the boundary of $\mathcal{M}$. The homomorphism ev is an isomorphism over the interior of $\mathcal{M}$, and its restriction to the boundary,

$$\mathrm{ev}_{\partial\mathcal{M}}: {}^bT_{\partial\mathcal{M}}\mathcal{M} \to T\partial\mathcal{M} \tag{2.1}$$

is surjective. Its kernel, a fortiori a rank-one bundle, is spanned by a canonical section denoted $\mathfrak{r}\partial_{\mathfrak{r}}$; $\mathfrak{r}$ refers to any smooth defining function for $\partial\mathcal{M}$ in $\mathcal{M}$, by convention positive in the interior of $\mathcal{M}$.

Since $C^\infty_{\tan}(\mathcal{M},T\mathcal{M})$ is closed under Lie brackets, there is an induced Lie bracket on $C^\infty(\mathcal{M};{}^bT\mathcal{M})$, as well as on the space of smooth sections of the complexification $\mathbb{C}{}^bT\mathcal{M}$ of ${}^bT\mathcal{M}$. So the notion of involutivity of a subbundle of $\mathbb{C}{}^bT\mathcal{M}$ is well defined.

Definition 2.2. A complex b-structure on $\mathcal{M}$ is an involutive subbundle

$${}^bT^{0,1}\mathcal{M} \subset \mathbb{C}{}^bT\mathcal{M}$$

such that

$${}^bT^{1,0}\mathcal{M} + {}^bT^{0,1}\mathcal{M} = \mathbb{C}{}^bT\mathcal{M} \tag{2.3}$$

and

$${}^bT^{1,0}\mathcal{M} \cap {}^bT^{0,1}\mathcal{M} = 0 \tag{2.4}$$

with ${}^bT^{1,0}\mathcal{M} = \overline{{}^bT^{0,1}\mathcal{M}}$. A complex b-manifold is a manifold with boundary together with a complex b-structure.

Thus a complex b-manifold is a complex manifold in the b-category. By the Newlander-Nirenberg Theorem [8], the interior of $\mathcal{M}$ is a complex manifold.

Complex b-structures, more generally, CR b-structures, were introduced in [4], some aspects of the boundary structure determined by a complex b-structure were analyzed in [5], and families of examples were presented in [6]. An in-depth study will appear in [7]. In particular, we showed in [5] that a complex b-structure on $\mathcal{M}$ determines the following on each component $\mathcal{N}$ of the boundary of $\mathcal{M}$:

(i) An involutive subbundle $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ with the property that $\mathcal{V} + \overline{\mathcal{V}} = \mathbb{C}T\mathcal{N}$.

(ii) A real vector field $\mathcal{T}$ on $\mathcal{N}$ such that $\mathcal{V} \cap \overline{\mathcal{V}} = \mathrm{span}_{\mathbb{C}}\{\mathcal{T}\}$.

Since $\overline{\mathcal{V}}$ is involutive, there is a natural complex

$$\cdots \to C^\infty(\mathcal{N};\textstyle\bigwedge^q\overline{\mathcal{V}}^*) \xrightarrow{\overline{\mathbb{D}}_q} C^\infty(\mathcal{N};\textstyle\bigwedge^{q+1}\overline{\mathcal{V}}^*) \to \cdots.$$

(iii) A family $\boldsymbol{\beta} \subset C^\infty(\mathcal{N};\overline{\mathcal{V}}^*)$ of $\overline{\mathbb{D}}$-closed sections of $\overline{\mathcal{V}}^*$ such that if $\beta \in \boldsymbol{\beta}$, then

$$\langle \beta, \mathcal{T}\rangle = a - i, \qquad a: \mathcal{N} \to \mathbb{R} \text{ smooth} \tag{2.5}$$

and an element $\beta' \in C^\infty(\mathcal{N};\overline{\mathcal{V}}^*)$ belongs to $\boldsymbol{\beta}$ if and only if there is $u: \mathcal{N} \to \mathbb{R}$ is smooth such that $\beta - \beta' = \overline{\mathbb{D}}u$.

It should be noted that $\overline{\mathcal{V}}$ is not a CR structure, but rather an elliptic structure because of (i) above (see Treves [11, 12] for the general definition of elliptic structure). However, for each $\beta \in \boldsymbol{\beta}$, $\ker \beta \subset \overline{\mathcal{V}}$ is a locally integrable (i.e., locally realizable) CR structure of hypersurface type. Local integrability is a consequence of a result of Nirenberg [9] which in this case gives that in a neighborhood of any point of $\mathcal{N}$ there are coordinates $x^1, \dots, x^{2n}, t$ such that $\overline{\mathcal{V}}$ is locally spanned by the vector fields

$$\partial_{x^i} + i\partial_{x^{i+n}},\ i = 1, \dots, n,\ \partial_t.$$

The functions $z^j = x^j + ix^{j+n}$, $j = 1, \dots, n$, are annihilated by the elements of $\overline{\mathcal{V}}$, hence also by the elements of $\ker \beta$ for any $\beta \in \boldsymbol{\beta}$.

The datum of a manifold $\mathcal{N}$, a vector subbundle $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$, a real vector field $\mathcal{T}$, and a $\overline{\mathbb{D}}$-closed element $\beta \in C^\infty(\mathcal{N}; \overline{\mathcal{V}}^*)$ satisfying (i)–(iii) permits the construction of a complex b-manifold whose boundary structure is the given one, namely, let

$$\mathcal{M} = [0, \infty) \times \mathcal{N}$$

with the fiber of ${}^bT^{0,1}\mathcal{M}$ at $(\mathfrak{r}, p)$ given by

$$T^{0,1}_{(\mathfrak{r},p)}\mathcal{M} = \{v + \langle \beta, v\rangle \mathfrak{r}\partial_{\mathfrak{r}} : v \in \overline{\mathcal{V}}_p\},$$

see [5, Proposition 2.6].

The principal aim of this paper is to present a classification theorem of such boundary structures assuming that $\mathcal{N}$ is compact and that there is a $\mathcal{T}$-invariant Riemannian metric g on $\mathcal{N}$. Under these hypotheses for a given component of ∂M one can show, [7], that there is an element $\beta \in \boldsymbol{\beta}$ such that (2.5) is improved to

$$\langle \beta, \mathcal{T}\rangle = a - i, \qquad a : \mathcal{N} \to \mathbb{R} \text{ constant.} \tag{2.6}$$

The number a is characteristic of the way the boundary structure is related to the complex b-structure. One then obtains a new structure by defining $\tilde{\beta} = -i(a - i)^{-1}\beta$, which of course has the property that $\langle \tilde{\beta}, \mathcal{T}\rangle = -i$ (that is, $\Re\langle \tilde{\beta}, \mathcal{T}\rangle = 0$), and $\tilde{\boldsymbol{\beta}}$ as the class of $\tilde{\beta}$ modulo $\overline{\mathbb{D}}(C^\infty(\mathcal{N}; R))$. The classification theorem in Section 4 concerns structures for which there is an element $\beta \in \boldsymbol{\beta}$ such that $\langle \tilde{\beta}, \mathcal{T}\rangle = -i$, and the result of [7] alluded to above guarantees no loss of generality. From the perspective of this work, which is just the classification result, the origin of the structure and the fact that there is β such that $\Re\langle \beta, \mathcal{T}\rangle$ is constant are immaterial.

3. Classification by relative Chern classes

Let $\mathcal{F}$ be the family of pairs $(\mathcal{N}, \mathcal{T})$ such that

1. $\mathcal{N}$ is a compact connected manifold without boundary;
2. $\mathcal{T}$ is a globally defined nowhere vanishing real vector field on $\mathcal{N}$; and
3. there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$.

The class $\mathcal{F}$ contains all pairs $(\mathcal{N},\mathcal{T})$ such that $\mathcal{N}$ is the circle bundle of a Hermitian line bundle over a compact connected manifold and $\mathcal{T}$ is the infinitesimal generator of the canonical S^1-action on $\mathcal{N}$.

We denote by $\mathfrak{a}_t$ the group of diffeomorphisms generated by $\mathcal{T}$, and by $\mathcal{O}_p$ the orbit of $\mathcal{T}$ through p. If $(\mathcal{N},\mathcal{T})$, $(\mathcal{N}',\mathcal{T}')\in\mathcal{F}$ and h is a smooth map from an open set of $\mathcal{N}'$ to one of $\mathcal{N}$ such that $h_*\mathcal{T}'=\mathcal{T}$, then h is called equivariant.

Definition 3.1. Two elements $(\mathcal{N},\mathcal{T})$, $(\mathcal{N}',\mathcal{T}')\in\mathcal{F}$ are globally equivalent (g-equivalent) if there is an equivariant diffeomorphism $h:\mathcal{N}'\to\mathcal{N}$. They are locally equivalent (l-equivalent) if there are open covers $\{U_a\}_{a\in A}$ of $\mathcal{N}$ and $\{U'_a\}_{a\in A}$ of $\mathcal{N}'$ by $\mathcal{T}$, resp. $\mathcal{T}'$-invariant open sets and equivariant diffeomorphisms $h_a:U'_a\to U_a$ for each $a\in A$ such that

$$h_a h_b^{-1}(p)\in\overline{\mathcal{O}}_p \quad \text{for every } a,\ b\in A \text{ and } p\in U_a\cap U_b. \tag{3.2}$$

Example 3.3. If $\mathcal{N}$ and $\mathcal{N}'$ are the respective circle bundles of Hermitian line bundles $E\to\mathcal{B}$ and $E'\to\mathcal{B}'$, and $(\mathcal{N},\mathcal{T})$ and $(\mathcal{N}',\mathcal{T}')$ locally equivalent, then the base spaces are diffeomorphic. If $(\mathcal{N},\mathcal{T})$ and $(\mathcal{N}',\mathcal{T}')$ are globally equivalent and $h:\mathcal{N}\to\mathcal{N}'$ is an equivariant diffeomorphism, then h induces a unitary line bundle isomorphism covering a diffeomorphism of the base spaces.

If $(\mathcal{N},\mathcal{T})\in\mathcal{F}$ define the relation $p\sim p'$ if and only if $p'\in\overline{\mathcal{O}}_p$. The fact that there is a $\mathcal{T}$-invariant metric implies that this is a relation of equivalence (the key fact being the implication $p'\in\overline{\mathcal{O}}_p \implies p\in\overline{\mathcal{O}}_{p'}$). Let $\mathcal{B}_\mathcal{N}$ be the quotient space. Then $\mathcal{B}_\mathcal{N}$, the base space of $(\mathcal{N},\mathcal{T})$, is a Hausdorff space. The following lemma is immediate in view of (3.2).

Lemma 3.4. *Let $(\mathcal{N},\mathcal{T})$, $(\mathcal{N}',\mathcal{T}')\in\mathcal{F}$ be locally equivalent. Then $\mathcal{B}_\mathcal{N}$ and $\mathcal{B}_{\mathcal{N}'}$ are homeomorphic.*

Henceforth we fix an element $(\mathcal{N},\mathcal{T})\in\mathcal{F}$, denote $\mathcal{B}_\mathcal{N}$ by $\mathcal{B}$ and let $\pi:\mathcal{N}\to\mathcal{B}$ be the quotient map. We aim first at giving a classification modulo global equivalence of the set of elements of $\mathcal{F}$ locally equivalent to $(\mathcal{N},\mathcal{T})$, by the elements of $H^2(\mathcal{B},\mathscr{Z})$ where $\mathscr{Z}$ is the sheaf of germs of locally constant functions on $\mathcal{B}$ with values in a certain free, finite rank abelian group $\mathfrak{z}$ to be described in a moment. This classification is similar to the classification of circle bundles (or complex line bundles) over $\mathcal{B}$ by their first Chern class.

Let $\operatorname{Homeo}(\mathcal{N})$ be the group of homeomorphisms $\mathcal{N}\to\mathcal{N}$ with the compact-open topology. The structure group of $(\mathcal{N},\mathcal{T})$ is the closure, to be denoted G, of the subgroup $\{\mathfrak{a}_t : t\in\mathbb{R}\}\subset\operatorname{Homeo}(\mathcal{N})$. It is clearly an abelian group. Fix a $\mathcal{T}$-invariant metric. Then we may view $\{\mathfrak{a}_t : t\in\mathbb{R}\}$ as a subgroup of the group of isometries of $\mathcal{N}$ with respect to the metric, a compact Lie group. Its closure is therefore compact.

Lemma 3.5. *The structure group G is a compact abelian Lie group acting on $\mathcal{N}$ by smooth diffeomorphisms, and is a subgroup of the group of isometries of $\mathcal{N}$ with respect to any $\mathcal{T}$-invariant metric.*

We will denote the action of an element $g \in G$ on $\mathcal{N}$ by $\mathfrak{A}_g$, and by $\mathfrak{A} : G \times \mathcal{N} \to \mathcal{N}$ the map $\mathfrak{A}(g, p) = \mathfrak{A}_g p$. Let $\mathfrak{g}$ be the Lie algebra of G, let $\exp : \mathfrak{g} \to G$ be the exponential map, and let $\mathfrak{z} \subset \mathfrak{g}$ be its kernel.

Definition 3.6. Let $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$, let $\mathcal{B}$ be its base space, and let $\pi : \mathcal{N} \to \mathcal{B}$ be the projection map. If $V \subset \mathcal{B}$, let $\mathcal{N}_V = \pi^{-1}(V)$.

1. For each open set $V \subset \mathcal{B}$ let $\mathcal{I}^\infty(\mathcal{N}_V)$ be the set of smooth $\mathcal{T}$-equivariant diffeomorphisms $h : \mathcal{N}_V \to \mathcal{N}_V$ such that $h(p) \in \overline{\mathcal{O}}_p$ for all $p \in \mathcal{N}_V$. It is easy to see that $\mathcal{I}^\infty(\mathcal{N}_V)$ is an abelian group under composition. The family $\{\mathcal{I}^\infty(\mathcal{N}_V)\}$ with the obvious restriction maps forms a presheaf over $\mathcal{B}$ giving an abelian sheaf $\mathscr{I}^\infty(\mathcal{N}) \to \mathcal{B}$.
2. If $V \subset \mathcal{B}$ is open, let $C^\infty(V, \mathfrak{g})$ be the space of smooth functions $\mathcal{N}_V \to \mathfrak{g}$ which are constant on the orbits of $\mathcal{T}$. These spaces also form an abelian presheaf. We let $\mathscr{C}^\infty(\mathcal{B}, \mathfrak{g}) \to \mathcal{B}$ be the corresponding sheaf.
3. Finally, let $\mathscr{Z} \to \mathcal{B}$ be the sheaf of locally constant $\mathfrak{z}$-valued functions on $\mathcal{B}$.

There is a natural map $\operatorname{Exp} : C^\infty(V, \mathfrak{g}) \to \mathcal{I}^\infty(\mathcal{N}_V)$, namely, if $f \in C^\infty(V, \mathfrak{g})$, let $\operatorname{Exp}(f)(p) = \mathfrak{A}_{\exp f(p)} p$. Since $f(\mathfrak{a}_t p) = f(p)$,

$$\operatorname{Exp}(f)(\mathfrak{a}_t p) = \mathfrak{A}_{\exp f(p)} \mathfrak{a}_t p = \mathfrak{a}_t \mathfrak{A}_{\exp f(p)} p.$$

So $\operatorname{Exp}(f)$ is an equivariant diffeomorphism (its inverse is $\operatorname{Exp}(-f)$). Thus $\operatorname{Exp}(f) \in \mathcal{I}^\infty(\mathcal{N}_V)$. With Exp we get a sheaf homomorphism

$$\operatorname{Exp} : \mathscr{C}^\infty(\mathcal{B}, \mathfrak{g}) \to \mathscr{I}^\infty(\mathcal{N}),$$

and with the inclusion $\mathfrak{z} \to \mathfrak{g}$, the sheaf homomorphism

$$\iota : \mathscr{Z} \to \mathscr{C}^\infty(\mathcal{B}, \mathfrak{g}).$$

Proposition 3.7. *The sequence*

$$0 \to \mathscr{Z} \xrightarrow{\iota} \mathscr{C}^\infty(\mathcal{B}, \mathfrak{g}) \xrightarrow{\operatorname{Exp}} \mathscr{I}^\infty(\mathcal{N}) \to 0 \tag{3.8}$$

is exact. The sheaf $\mathscr{C}^\infty(\mathcal{B}, \mathfrak{g})$ is a fine sheaf, so the long exact sequence in cohomology gives an isomorphism $H^1(\mathcal{B}, \mathscr{I}^\infty(\mathcal{N})) \to H^2(\mathcal{B}, \mathscr{Z}) \approx H^2(\mathcal{B}, \mathbb{Z}^d)$ where d is the dimension of G.

The proofs of this proposition and the next theorem are given below.

Theorem 3.9. *Let $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$. There is a natural one-to-one correspondence between the elements of $H^2(\mathcal{B}, \mathscr{Z})$ and the g-equivalence classes of elements of $\mathcal{F}$ which are locally equivalent to $\mathcal{N}$.*

If $(\mathcal{N}', \mathcal{T}')$ is locally equivalent to $(\mathcal{N}, \mathcal{T})$ we write $\mathrm{c}_1(\mathcal{N}', \mathcal{N})$ for the element of $H^2(\mathcal{B}, \mathbb{Z})$ corresponding to $(\mathcal{N}', \mathcal{T}')$. The element $\mathrm{c}_1(\mathcal{N}', \mathcal{N})$ should be regarded as a relative Chern class.

Example 3.10. Suppose that $\mathcal{B}$ is a smooth manifold and $\pi : \mathcal{N} \to \mathcal{B}$ is the circle bundle of a Hermitian line bundle $E \to \mathcal{B}$. Let $E' \to \mathcal{B}$ be another Hermitian line bundle with circle bundle $\pi' : \mathcal{N}' \to \mathcal{B}$. Let $\{V_a\}_{a \in A}$ be an open cover of $\mathcal{B}$ such that for each a, $E|_{V_a}$ is isomorphic to $E'|_{V_a}$ with unitary isomorphisms h_a covering the identity map on V_a. Let $U_a = \mathcal{N}_{V_a} = \pi^{-1}(V_a)$, $U'_a = \mathcal{N}'_{V_a}$. The h_a restrict to equivariant diffeomorphisms $h_a : U'_a \to U_a$. Let

$$h_{ab} = h_a \circ h_b^{-1} : U_a \cap U_b \to U_a \cap U_b.$$

The h_{ab} determine the element $\mathrm{c}_1(\mathcal{N}', \mathcal{N}) \in H^2(\mathcal{B}, \mathbb{Z})$. It is not hard to show that $\mathrm{c}_1(\mathcal{N}', \mathcal{N}) = \mathrm{c}_1(E') - \mathrm{c}_1(E)$.

The proofs of Proposition 3.7 and Theorem 3.9 require some preparation. The following theorem yields explicit information that will be useful in the development of the theory.

Theorem 3.11. *Suppose there is a $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$. Then there is a smooth embedding $F : \mathcal{N} \to \mathbb{C}^N$ with image contained in the sphere S^{2N-1}, none of whose component functions vanishes to infinite order at any point, such that for some positive numbers τ_j, $j = 1, \dots N$,*

$$F_*\mathcal{T} = i \sum_j \tau_j (w^j \partial_{w^j} - \overline{w}^j \partial_{\overline{w}^j}) \tag{3.12}$$

in the coordinates $w^1, \dots, w^N$ of $\mathbb{C}^N$.

The proof relies on an idea originally due to Bochner [1], expressed in this case by judiciously choosing enough functions $\tilde{F}^\ell$ that are at the same time eigenfunctions of $\mathcal{T}$ and eigenfunctions of the Laplacian with respect to some fixed $\mathcal{T}$-invariant Riemannian metric on $\mathcal{N}$, and then using them to construct the map F. The details are given in [7].

Note that the manifold S^{2N-1} together with the vector field $\mathcal{T}'$ on S^{2N-1} given by the expression on the right in (3.12) is an example of a pair in the class $\mathcal{F}$. The standard metric on S^{2N-1} is $\mathcal{T}$-invariant. Observe in passing that the orbits of $\mathcal{T}'$ need not be compact.

Recall that G is the closure of $\{\mathfrak{a}_t\}$ in the compact-open topology of Homeo($\mathcal{N}$). Fix a map $F : \mathcal{N} \to S^{2N-1}$ having the properties stated in Theorem 3.11 and let $\mathcal{T}'$ again be the vector field on S^{2N-1} given by the expression on the right in (3.12). Let $\mathfrak{a}'_t$ be the one-parameter group generated by $\mathcal{T}'$. The set

$$G_0 = \text{ closure of } \{(e^{i\tau_1 t}, \dots, e^{i\tau_N t}) \in S^1 \times \cdots \times S^1 : t \in \mathbb{R}\} \tag{3.13}$$

is the structure group of the pair $(S^{2N-1}, \mathcal{T}')$. For any $w \in S^{2N-1}$, the closure of the orbit of w by $\mathcal{T}'$ is

$$\{(\omega^1 w^1, \dots, \omega^N w^N) : (\omega^1, \dots, \omega^N) \in G_0\}.$$

Let

$$W = \{p : F^\ell(p) \neq 0,\ \ell = 1, \dots, N\}.$$

Then

> W is open and dense in $\mathcal{N}$, if $\{t_\nu\}_{\nu=1}^\infty$ is such that $\mathfrak{a}_{t_\nu} p$ converges for some $p \in W$, then $\{\mathfrak{a}_{t_\nu}\}_{\nu=1}^\infty$ converges in the C^∞ topology, and if $p \in W$ and $\mathfrak{A}_g p = p$, then g is the identity. (3.14)

Indeed, if $\mathfrak{a}_{t_\nu} p$ converges, then so does $\mathfrak{a}'_{t_\nu} F(p)$ since F is equivariant. Since $F^\ell(p) \neq 0$ for each ℓ, $\{e^{i\tau_\ell t_\nu}\}_{\nu=1}^\infty$ converges for each ℓ. It follows that $\mathfrak{a}'_{t_\nu} : S^{2N-1} \to S^{2N-1}$ converges in the C^∞ topology, and then so does $\mathfrak{a}_{t_\nu} = F^{-1} \circ \mathfrak{a}'_{t_\nu} \circ F$.

Now, if $g \in G$, then there is a sequence $\{t_\nu\}_{\nu=1}^\infty$ such that $\mathfrak{a}_{t_\nu} \to g$ in the compact-open topology. In particular, if $p \in W$, then $\mathfrak{a}_{t_\nu} p$ converges, so $\mathfrak{a}_{t_\nu} \to g$ in the C^∞ topology.

The elements $g = \lim \mathfrak{a}_{t_\nu} \in G$ and $\lim(e^{i\tau_1 t_\nu}, \dots, e^{i\tau_N t_\nu}) = \omega \in G_0$ are related by $F(\mathfrak{A}_g p) = \mathfrak{A}'_\omega F(p)$. If $g \in G$ has the property that there is $p \in W$ such that $\mathfrak{A}_g p = p$, then $\mathfrak{A}'_\omega F(p) = F(p)$, so $\omega = I$ and $\mathfrak{A}_g$ is the identity map. Thus, tautologically, g is the identity element of G.

Let $\mathcal{J}_p$ denote the isotropy subgroup of G at p. We now show that the set

$$\mathcal{N}_{\text{reg}} = \{p \in \mathcal{N} : \mathcal{J}_p \text{ is trivial}\}.$$

has the properties listed for W in (3.14). One of the virtues of $\mathcal{N}_{\text{reg}}$ is that it is independent of the auxiliary map F.

Lemma 3.15. *The set $\mathcal{N}_{\text{reg}}$ is open, dense, and G-invariant. If $p \in \mathcal{N}_{\text{reg}}$ and $\{t_\nu\}_{\nu=1}^\infty$ is a sequence such that $\mathfrak{a}_{t_\nu} p$ converges, then $\mathfrak{a}_{t_\nu}$ converges in G in the C^∞ topology. If $g \in G$ and $\mathfrak{A}_g p = p$ for some $p \in \mathcal{N}_{\text{reg}}$, then g is the identity.*

Proof. The last property is the definition of $\mathcal{N}_{\text{reg}}$. The G-invariance of $\mathcal{N}_{\text{reg}}$ is clear. Fix some $\mathcal{T}$-invariant metric on $\mathcal{N}$. Let $p_0 \in \mathcal{N}$ be arbitrary. Let $\pi : \mathcal{H}_{\overline{\mathcal{O}}_{p_0}} \to \overline{\mathcal{O}}_{p_0}$ be the orthogonal bundle of $T\overline{\mathcal{O}}_{p_0}$ in $T\mathcal{N}$ with respect to the metric, let

$$B_{\overline{\mathcal{O}}_{p_0},\varepsilon} = \{v \in \mathcal{H}_{\overline{\mathcal{O}}_{p_0}} : |v| < \varepsilon\}$$

and let

$$U_{\overline{\mathcal{O}}_{p_0},\varepsilon} = \exp(B_{\overline{\mathcal{O}}_{p_0},\varepsilon})$$

where, as usual, exp maps a sufficiently small vector $v \in T\mathcal{N}$ to the image of 1 by the geodesic through v. Since $\overline{\mathcal{O}}_{p_0}$ is compact, there is $\varepsilon > 0$ such that for $B = B_{\overline{\mathcal{O}}_{p_0},\varepsilon}$ the map $\exp|_B : B \to \mathcal{N}$ is a diffeomorphism onto its image $U = U_{\overline{\mathcal{O}}_{p_0},\varepsilon}$. Since $U = \{p \in \mathcal{N} : \operatorname{dist}(p, \overline{\mathcal{O}}_{p_0}) < \varepsilon\}$, U is $\mathcal{T}$-invariant. Let $\rho = \pi \circ (\exp|_B)^{-1}$. If $g \in G$, then $\mathfrak{A}_g$ is an isometry, so $\rho\mathfrak{A}_g = \mathfrak{A}_g\rho$ in U. It follows that if $g \in \mathcal{J}_p$ with $p \in U$, $\rho(p) = p_0$, then $g \in \mathcal{J}_{p_0}$. Consequently, if $p_0 \in \mathcal{N}_{\text{reg}}$, then $U \subset \mathcal{N}_{\text{reg}}$. Thus $\mathcal{N}_{\text{reg}}$ is open. Note in passing that if $p_0 \in \mathcal{N}_{\text{reg}}$, then $\rho|_{\overline{\mathcal{O}}_p} : \overline{\mathcal{O}}_p \to \overline{\mathcal{O}}_{p_0}$ is a diffeomorphism.

The set W used in the proof of Lemma 3.5 is a subset of $\mathcal{N}_{\text{reg}}$. Since W is dense, so is $\mathcal{N}_{\text{reg}}$.

Finally, let $p_0 \in \mathcal{N}_{\text{reg}}$ and suppose that $\{t_\nu\}_{\nu=1}^\infty$ is a sequence such that $\mathfrak{a}_{t_\nu} p_0$ converges. Since W is dense in $\mathcal{N}$, there is $p \in U_{\overline{\mathcal{O}}_{p_0},\varepsilon} \cap W$. Since $\rho|_{\overline{\mathcal{O}}_p} : \overline{\mathcal{O}}_p \to \overline{\mathcal{O}}_{p_0}$ is a diffeomorphism that commutes with each $\mathfrak{a}_t$, $\mathfrak{a}_{t_\nu} p$ converges. Therefore $\mathfrak{a}_{t_\nu}$ converges, by (3.14). □

Let $\mathcal{B}_{\text{reg}} = \pi(\mathcal{N}_{\text{reg}})$. One can show that $\mathcal{B}_{\text{reg}}$ is a smooth manifold and that $\mathcal{N}_{\text{reg}} \to \mathcal{B}_{\text{reg}}$ is a principal G-bundle.

A variant of the proof of Lemma 3.5 gives:

Lemma 3.16. *Let $(\mathcal{N}',\mathcal{T}')\in\mathcal{F}$ be locally equivalent to $(\mathcal{N},\mathcal{T})$, and let $G_{\mathcal{N}}$, $G_{\mathcal{N}'}$ be the respective structure groups. Then there is a group isomorphism*

$$\Psi_{\mathcal{N},\mathcal{N}'}:G_{\mathcal{N}'}\to G_{\mathcal{N}}$$

such that for any open $\mathcal{T}'$-invariant set $U'\subset\mathcal{N}'$ and $\mathcal{T}$-invariant set $U\subset\mathcal{N}$, if $h:U'\to U$ is an equivariant diffeomorphism, then

$$h\circ\mathfrak{A}'_{g'}=\mathfrak{A}_{\Psi_{\mathcal{N},\mathcal{N}'}g'}\circ h.$$

Proof. Pick some equivariant diffeomorphism $h:U'\to U$. If $g'\in G_{\mathcal{N}'}$, then there is a sequence $\{t_\nu\}$ such that $\mathfrak{a}'_{t_\nu}\to g$ in the C^∞ topology. If

$$p'\in U'\cap\mathcal{N}'_{\mathrm{reg}}\cap h^{-1}(U\cap\mathcal{N}_{\mathrm{reg}}),$$

then the continuity and equivariance of h and the convergence of $\mathfrak{a}'_{t_\nu}p$ give the convergence of $\mathfrak{a}_{t_\nu}h(p)$, hence by Lemma 3.15 the convergence of $\mathfrak{a}_{t_\nu}$ to some element $\Psi_{\mathcal{N},\mathcal{N}'}g'$. It is easy to verify that Ψ is a group isomorphism (the inverse being $\Psi_{\mathcal{N}',\mathcal{N}}$) independent of the auxiliary equivariant diffeomorphism used to define it. □

The following lemma is the key component in the proof of the surjectivity of Exp in the sequence (3.8)

Lemma 3.17. *Let $U\subset\mathbb{R}^n$ be open, let f, $g:U\to\mathbb{C}$ be smooth and such that $|f|=|g|$ and f, g are not flat at any point of U. Then there is a unique smooth function $\omega:U\to S^1\subset\mathbb{C}$ such that $f=\omega g$.*

Proof. On the open set $V=\{x\in U: f(x)\neq 0 \text{ and } g(x)\neq 0\}$ we have that f/g is a smooth function with values in S^1. So, since V is dense, ω is unique if it exists. Let $x_0\in U$ be arbitrary. Since neither of the functions f or g is flat at x_0, the Malgrange Preparation Theorem gives that there are coordinates $(y_1,\dots,y_n)$ centered at x_0 such that with $y'=(y_1,\dots,y_{n-1})$ we have

$$qf=y_n^k+\sum_{\ell=0}^{k-1}r_\ell(y')y_n^\ell,\quad q'g=y_n^{k'}+\sum_{\ell=0}^{k'-1}r'_\ell(y')y_n^\ell,$$

near x_0 with smooth functions q, q', r_ℓ, r'_ℓ, and $q(x_0)$, $q'(x_0)\neq 0$. The condition $|f|=|g|$ gives that the polynomials $y_n^k+\sum_{\ell=0}^{k-1}r_\ell(y')y_n^\ell$, $y_n^{k'}+\sum_{\ell=0}^{k'-1}r'_\ell(y')y_n^\ell$ have the same roots for each fixed y'. Since f/g and g/f are both bounded on V and V is dense, these roots appear with the same multiplicity in both polynomials, that is, they are the same polynomial. Consequently $qf=q'g$ and thus $f=\omega g$ near x_0 with $\omega=q/q'$ smooth. □

Surjectivity of Exp in (3.8) is an immediate consequence of:

Lemma 3.18. *Let $V\subset\mathcal{B}$ be open and let $h\in\mathcal{I}^\infty(\mathcal{N}_V)$. For every $x_0\in V$ there are a neighborhood $V'\subset V$ of x_0 and $f\in C^\infty(V',\mathfrak{g})$ such that $h=\mathrm{Exp}(f)$ in $\mathcal{N}_{V'}$.*

Proof. Let $F : \mathcal{N} \to S^{2N-1}$ be a map as in the proof of Theorem 3.11, let $\mathcal{T}'$ be the vector field (3.12) and let $\mathfrak{a}'_t$ denote the associated one parameter group it generates; recall that no component of F^ℓ vanishes to infinite order at any point of $\mathcal{N}$. Suppose $h \in \mathcal{I}^\infty(\mathcal{N}_V)$. If $p \in \mathcal{N}_V$, then $F(h(p))$ is an element in the closure of the orbit of $F(p)$, so there is $\omega(p) \in G_0$, the structure group of $(S^{2N-1}, \mathcal{T}')$ (see (3.13)) such that $F(h(p)) = \mathfrak{A}'_{\omega(p)}F(p)$; $\omega(p)$ need not be unique. Componentwise this gives $F^\ell \circ h = \omega^\ell F^\ell$. In particular $|F^\ell \circ h| = |F^\ell|$. Since neither F^ℓ nor $F^\ell \circ h$ vanish to infinite order at any point, Lemma 3.17 gives that ω^ℓ is smooth. Let $g(p) \in G$ be the element that corresponds to $\omega(p)$ via the isomorphism of Lemma 3.16. So $g : \mathcal{N}_V \to G$ is smooth and $h(p) = \mathfrak{A}_{g(p)}p$. Furthermore, g is constant on the orbits of $\mathfrak{a}_t$. Indeed, on the one hand

$$h(\mathfrak{a}_t p) = \mathfrak{A}_{g(\mathfrak{a}_t p)}\mathfrak{a}_t p = \mathfrak{a}_t \mathfrak{A}_{g(\mathfrak{a}_t p)} p$$

and on the other

$$h(\mathfrak{a}_t p) = \mathfrak{a}_t h(p) = \mathfrak{a}_t \mathfrak{A}_{g(p)} p,$$

so $\mathfrak{A}_{g(\mathfrak{a}_t p)}p = \mathfrak{A}_{g(p)}p$ for any $p \in \mathcal{N}_V$. Then $g(\mathfrak{a}_t p) = g(p)$ if $p \in \mathcal{N}_V \cap \mathcal{N}_{\text{reg}}$, so $g \circ \mathfrak{a}_t = g$ by continuity of g and density of $\mathcal{N}_{\text{reg}}$. Given $p_0 \in \mathcal{N}_V$, let $\rho : U \to \overline{\mathcal{O}}_{p_0}$ be a tubular neighborhood map as in the proof of Lemma 3.15, contained in $\mathcal{N}_V$. Consider now the problem of lifting $g : U \to G$ to a map $f : U \to \mathfrak{g}$,

$$\begin{array}{ccc} & & \mathfrak{g} \\ & & \big\downarrow{\scriptstyle \exp} \\ U & \xrightarrow[g]{} & G \end{array}$$

Since U is contractible to $\overline{\mathcal{O}}_{p_0}$ and g is constant on $\overline{\mathcal{O}}_{p_0}$, g maps the fundamental group of U to the identity element of the fundamental group of G. So there is $f : U \to \mathfrak{g}$ such that $\exp \circ f = g|_U$. Let $V' = \pi(U)$, so $\mathcal{N}_{V'} = U$. If $p \in U$, then $\exp(f(\mathfrak{a}_t p)) = g(\mathfrak{a}_t p) = \mathfrak{g}(p) = \exp(f(p))$ gives that $\exp(f(\mathfrak{a}_t p) - f(p))$ is the identity element of G for any $t \in \mathbb{R}$. So $t \mapsto f(\mathfrak{a}_t p) - f(p)$ is a continuous $\mathfrak{z}$-valued function that vanishes at $t = 0$. Hence $f(\mathfrak{a}_t p) = f(p)$ for every $t \in \mathbb{R}$ and $p \in U$. Consequently $f \in C^\infty(V, \mathfrak{g})$ and $h = \operatorname{Exp}(f)$ on $\mathcal{N}_{V'}$. $\square$

The fact that $\mathscr{C}^\infty(\mathcal{B}, \mathfrak{g}) \to \mathcal{B}$ is a fine sheaf will follow from existence of partitions of unity in $C^\infty(\mathcal{B}, \mathbb{R})$ (recall that this is the space of smooth functions on $\mathcal{N}$ that are constant on orbits of $\mathcal{T}$).

Let μ denote the normalized Haar measure of G.

Lemma 3.19. *Suppose $V \subset \mathcal{B}$ is open. For $f \in C^\infty(\mathcal{N}_V)$ define $\operatorname{Av} f : \mathcal{N}_V \to \mathbb{C}$ by*

$$\operatorname{Av} f(p) = \int f(g \cdot p) d\mu(g).$$

Then $\operatorname{Av} f$ is smooth. If $f \in C^\infty_c(\mathcal{N}_V)$, then $\operatorname{Av} f \in C^\infty_c(\mathcal{N}_V)$.

Proof. Let $\rho : G \times \mathcal{N} \to \mathcal{N}$ be the canonical projection. If $g \in C^\infty(G \times \mathcal{N}_V)$ then $\rho_*(g\mu)$ is smooth. If $f \in C^\infty(\mathcal{N}_V)$, then $\mathfrak{A}^* f \in C^\infty(G \times \mathcal{N}_V)$, so $\rho_*((\mathfrak{A}^* f)\mu)$ is smooth. But $\rho_*((\mathfrak{A}^* f)\mu) = \operatorname{Av} f$. The proof of the last statement is also elementary. $\square$

As a consequence we have existence of smooth locally finite partitions of unity for $\mathcal{B}$.

Corollary 3.20. *Let $\{V_a\}$ be an open cover of $\mathcal{B}$. Then there is a family $\{\chi_\gamma\}_{\gamma\in\Gamma}$ of nonnegative functions $\chi_\gamma \in C^\infty(\mathcal{B})$ such that for every γ there is $a(\gamma)$ such that $\operatorname{supp}\chi_\gamma \subset \mathcal{N}_{V_{a(\gamma)}}$, for every $K \Subset \mathcal{B}$ the set $\{\gamma : \operatorname{supp}\chi_\gamma \cap \mathcal{N}_K\}$ is finite, and $\sum_\gamma \chi_\gamma = 1$.*

A partition of unity as in the corollary gives a partition of unity for $\mathscr{C}^\infty(\mathcal{B}, \mathfrak{g})$, so the latter is a fine sheaf.

Proof of Proposition 3.7. Let $V \subset \mathcal{B}$ be open. The map ι sends a locally constant function $\nu : V \to \mathfrak{z}$ to the element $\nu \circ \pi$ of $C^\infty(V, \mathfrak{g})$, so ι is injective.

To see exactness at $\mathscr{C}^\infty(\mathcal{B}; \mathfrak{g})$, suppose $\nu : V \to \mathfrak{z}$ is locally constant. Then $\nu \circ \pi \in C^\infty(V, \mathfrak{g})$ and $\exp(\nu \circ \pi)$ is the identity. So $\iota(\nu \circ \pi) \in \ker \operatorname{Exp}$. Suppose now that $f \in C^\infty(V, \mathfrak{g})$ and $\operatorname{Exp}(f) \in \mathcal{I}^\infty(\mathcal{N}_V)$ is the identity: $\mathfrak{A}_{\exp f(p)} p = p$ for all $p \in \mathcal{N}_V$, in particular if $p \in \mathcal{N}_V \cap \mathcal{N}_{\mathrm{reg}}$. By Lemma 3.15, $\exp f(p)$ is the identity element of G when $p \in \mathcal{N}_V \cap \mathcal{N}_{\mathrm{reg}}$. So $f|_{\mathcal{N}_{\mathrm{reg}} \cap \mathcal{N}_V}$ has values in $\mathfrak{z}$. Since f is smooth and $\mathcal{N}_{\mathrm{reg}}$ is dense, f has values in $\mathfrak{z}$, and since f is constant on fibers, there is a locally constant function $\nu : V \to \mathfrak{z}$ such that $f = \nu \circ \pi$.

Finally, Lemma 3.18 gives that Exp in (3.8) is surjective. $\square$

Proof of Theorem 3.9. Let $\mathbf{h} \in H^1(\mathscr{I}^\infty(\mathcal{N}))$. We will, essentially by following the proof of the corresponding statement for line bundles, show that there is a well-defined g-equivalence class of elements associated with $\mathbf{h}$. To construct a representative of this class, choose an open cover $\{V_a\}_{a\in A}$ of $\mathcal{B}$ such that $\mathbf{h}$ is represented by a cocycle $\{h_{ab} \in \mathcal{I}^\infty(\mathcal{N}_{V_a \cap V_b})\}$. Thus h_{aa} is the identity map, $h_{ab} = h_{ba}^{-1}$, and $h_{ab} h_{bc} h_{ca} = I$. We then get a manifold

$$\mathcal{N}' = \bigsqcup_{a\in A} \mathcal{N}_{V_a} / \sim$$

as for vector bundles, where $\sim$ is the relation of equivalence on $\bigsqcup_{a\in A} \mathcal{N}_{V_a}$ for which $p \in \mathcal{N}_{V_a}$ and $q \in \mathcal{N}_{V_b}$ are equivalent if and only $p, q \in \mathcal{N}_{V_a \cap V_b}$ and $p = h_{ab} q$. The set of equivalence classes of elements of $\mathcal{N}_{V_a}$ is an open set $\mathcal{N}'_{V_a} \subset \mathcal{N}'$, and if $p' \in \mathcal{N}'_{V_a}$ then there is a unique $p \in \mathcal{N}_{V_a}$ such that the class of p is p'. This gives a map $h_a : \mathcal{N}'_{V_a} \to \mathcal{N}_{V_a}$ for each a, a diffeomorphism. By the definition of the relation of equivalence, if $V_a \cap V_b \neq \emptyset$ then $h_a h_b^{-1} = h_{ab}$ on $\mathcal{N}_{V_a \cap V_b}$. In particular (3.2) is satisfied. Let $\mathcal{T}_a = dh_a^{-1} \mathcal{T}$. Then $\mathcal{T}_b = \mathcal{T}_a$ on $\mathcal{N}'_{V_a} \cap \mathcal{N}'_{V_b}$, since $dh_{ab} \mathcal{T} = \mathcal{T}$. Thus $\mathcal{N}'$ admits a global nowhere vanishing vector field $\mathcal{T}'$, and by definition $dh_a \mathfrak{a}'_t = \mathfrak{a}_t h_a$ for each $a \in A$, i.e., h_a is equivariant. In particular, the orbits of $\mathcal{T}'$ are compact. By construction, $(\mathcal{N}', \mathcal{T}')$ is locally equivalent to $(\mathcal{N}, \mathcal{T})$, and one can construct a

$\mathcal{T}'$-invariant metric on $\mathcal{N}'$ using one such for $\mathcal{N}$, the maps h_a, and a partition of unity as in Corollary 3.20. So $(\mathcal{N}', \mathcal{T}') \in \mathcal{F}$ is locally equivalent to $(\mathcal{N}, \mathcal{T})$.

If $\{\tilde{h}_{ab} \in \mathcal{I}^\infty(\mathcal{N}_{V_a \cap V_b})\}$ also represents $\mathbf{h}$, let $(\tilde{\mathcal{N}}', \tilde{\mathcal{T}}')$ be the element of $\mathcal{F}$ constructed as above with the $\tilde{h}_{ab}$. We will show that $\mathcal{N}'$ is globally equivalent to $\tilde{\mathcal{N}}'$. Let $\tilde{h}_a : \tilde{\mathcal{N}}'_{V_a} \to \mathcal{N}_{V_a}$ be maps such that $\tilde{h}_{ab} = \tilde{h}_a \tilde{h}_b^{-1}$. Because $\{h_{ab}\}$ and $\{\tilde{h}_{ab}\}$ represent the same element $\mathbf{h}$, there is, perhaps after passing to a refinement of the cover $\{V_a\}$, an element $g_a \in \mathcal{I}^\infty(\mathcal{N}_{V_a})$ for each $a \in A$ such that

$$\tilde{h}_{ab} = h_{ab} g_a g_b^{-1}. \tag{3.21}$$

Passing to a further refinement of the cover $\{V_a\}$ and using Lemma 3.18 we may assume that for each a there is $f_a \in C^\infty(V_a, \mathfrak{g})$ such that $g_a = \operatorname{Exp} f_a$. Let $\Psi : G' \to G$ be the isomorphism of Lemma 3.16. We have

$$g_a h_a(p') = \mathfrak{A}_{\exp f_a(h_a(p'))} h_a(p') = h_a(\mathfrak{A}_{\Psi^{-1} \exp f_a(h_a(p'))} p'), \quad p' \in \mathcal{N}_{V_a},$$

in other words, $g_a h_a = h_a g'_a$ with $g'_a(p') = \operatorname{Exp}(d\Psi^{-1}(f_a \circ h_a))$. From (3.21) we get

$$\tilde{h}_a \tilde{h}_b^{-1} = g_a h_a h_b^{-1} g_b^{-1} = h_a g'_a (h_b g'_b)^{-1}$$

so

$$\tilde{h}_b^{-1} h_b g'_b = \tilde{h}_a^{-1} h_a g'_a.$$

That is, the maps $\tilde{h}_a^{-1} h_a g'_a : \mathcal{N}'_{V_a} \to \tilde{N}'_{V_a}$, which are equivariant diffeomorphisms, give a global map $\mathcal{N}' \to \tilde{\mathcal{N}}'$. So there is a well-defined g-equivalence class of elements associated with $\mathbf{h}$.

Conversely, given an element $(\mathcal{N}', \mathcal{T}') \in \mathcal{F}$ which is locally equivalent to $(\mathcal{N}, \mathcal{T})$ by way of maps $h_a : U'_a \to U_a$ as in Definition 3.1 we get a cocycle $h_{ab} = h_a h_b^{-1}$, $h_{ab} \in \mathcal{I}^\infty(\mathcal{N}_{V_a \cap V_b})$. It is not hard to verify that the element $\mathbf{h} \in H^1(\mathscr{I}^\infty(\mathcal{N}))$ constructed from this cocycle as above is globally equivalent to $(\mathcal{N}', \mathcal{T}')$. □

4. Classification by a Picard group

We now present an analogue of the classification of holomorphic line bundles over compact complex manifolds by the Picard group.

Let $\mathcal{F}_{\text{ell}}$ be the set of triples $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$ such that $(\mathcal{N}, \mathcal{T}) \in \mathcal{F}$ and $\overline{\mathcal{V}} \subset \mathbb{C}T\mathcal{N}$ is an elliptic structure with $\mathcal{V} \cap \overline{\mathcal{V}} = \operatorname{span}_{\mathbb{C}} \mathcal{T}$ that admits a $\overline{\mathbb{D}}$-closed element $\beta \in C^\infty(\mathcal{N}, \overline{\mathcal{V}}^*)$ such that $\mathbf{i}_{\mathcal{T}} \beta = -i$.

Definition 4.1. Two elements $(\mathcal{N}, \mathcal{T}, \overline{\mathcal{V}})$, $(\mathcal{N}', \mathcal{T}', \overline{\mathcal{V}}') \in \mathcal{F}_{\text{ell}}$ will be said to be globally ell-equivalent if there is an equivariant diffeomorphism $h : \mathcal{N}' \to \mathcal{N}$ such that $h_* \overline{\mathcal{V}}' = \overline{\mathcal{V}}$. They are locally ell-equivalent if there are open covers $\{U_a\}_{a \in A}$ of $\mathcal{N}$ and $\{U'_a\}_{a \in A}$ of $\mathcal{N}'$ by $\mathcal{T}$, resp. $\mathcal{T}'$-invariant open sets and equivariant diffeomorphisms $h_a : U'_a \to U_a$ for each $a \in A$ such that $h_a h_b^{-1}$ satisfies (3.2) and preserves $\overline{\mathcal{V}}$.

Fix $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})\in\mathcal{F}_{\text{ell}}$. Let $V\subset\mathcal{B}$ be open. An element $h\in\mathcal{I}^\infty(\mathcal{N}_V)$ preserves $\overline{\mathcal{V}}$ if $h_*(\overline{\mathcal{V}}_{\mathcal{N}_V})\subset\overline{\mathcal{V}}_{\mathcal{N}_V}$.

1. For each open $V\subset\mathcal{B}$ let $\mathcal{I}^{\overline{\mathcal{V}}}(\mathcal{N}_V)$ be the subgroup of $\mathcal{I}^\infty(\mathcal{N}_V)$ whose elements h preserve $\overline{\mathcal{V}}$. The associated sheaf is $\mathscr{I}^{\overline{\mathcal{V}}}(\mathcal{N})$.
2. For each open $V\subset\mathcal{B}$ let $C^{\overline{\mathcal{V}}}(V,\mathfrak{g})$ be the subspace of $C^\infty(V,\mathfrak{g})$ whose image by Exp is $\mathcal{I}^{\overline{\mathcal{V}}}(\mathcal{N}_V)$. The associated sheaf is denoted $\mathscr{C}^{\overline{\mathcal{V}}}(\mathcal{B},\mathfrak{g})$.

Clearly, the sequence

$$0\to\mathscr{Z}\to\mathscr{C}^{\overline{\mathcal{V}}}(\mathcal{B},\mathfrak{g})\to\mathscr{I}^{\overline{\mathcal{V}}}(\mathcal{N})\to 0$$

is exact.

Theorem 4.2. *Let $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})\in\mathcal{F}_{\text{ell}}$. There is a natural one-to-one correspondence between the elements of $H^1(\mathcal{B},\mathscr{I}^{\overline{\mathcal{V}}}(\mathcal{N})))$ and the global* ell*-equivalence classes of elements of $\mathcal{F}_{\text{ell}}$ which are locally* ell*-equivalent to $\mathcal{N}$.*

Proof. Let $\mathbf{h}\in H^1(\mathscr{I}(\mathcal{N}))$, let $\{V_a\}_{a\in A}$ be an open cover of $\mathcal{B}$, and suppose that the Čech cocycle $\{h_{ab}\in\mathcal{I}^{\overline{\mathcal{V}}}(\mathcal{N}_{V_a\cap V_b})\}$ represents $\mathbf{h}$. Let $(\mathcal{N}',\mathcal{T}')\in\mathcal{F}$ be the element constructed in the proof of Theorem 3.9 using $\{h_{ab}\}$, let $h_a:\mathcal{N}'_{V_a}\to\mathcal{N}_{V_a}$ be the corresponding maps. Since $h_ah_b^{-1}$ preserves $\overline{\mathcal{V}}$ for each a, b, $\mathcal{N}'$ inherits an elliptic structure $\overline{\mathcal{V}}'$, $\overline{\mathcal{V}}'=h_{a*}^{-1}\overline{\mathcal{V}}$, such that $\mathcal{V}'\cap\overline{\mathcal{V}}'=\operatorname{span}_{\mathbb{C}}\mathcal{T}'$.

The kernel $\overline{\mathcal{K}}\subset\overline{\mathcal{V}}$ of β is a CR structure of CR codimension 1. Let θ be the real one-form that vanishes on $\mathcal{K}\oplus\overline{\mathcal{K}}$ and satisfies $\langle\theta,\mathcal{T}\rangle=1$. Since $\overline{\mathbb{D}}\beta=0$ and $\mathbf{i}_{\mathcal{T}}\beta$ is constant, $\mathbf{i}_{\mathcal{T}}d\theta=0$. Let $\{\chi_a\}_{a\in A}\subset C^\infty(\mathcal{B})$ be a locally finite partition of unity subordinate to the cover $\{V_a\}$, as provided by Corollary 3.20. Since only finitely many of the χ_a are nonzero (because $\mathcal{N}$ is compact) there is no loss of generality if we assume that already the index set for the partition is the same as that for the cover and that $\operatorname{supp}\chi_a\subset\mathcal{N}_{V_a}$ for each a. Let $\theta'=\sum\chi_ah_a^*\theta$. Then $\iota_{\mathcal{T}'}\theta'=1$ and $\iota_{\mathcal{T}'}d\theta'=0$. It follows that the restriction $i\beta'$ of θ' to $\overline{\mathcal{V}}$ is $\overline{\mathbb{D}}$-closed which of course satisfies $i\mathbf{i}_{\mathcal{T}'}\beta'=1$. Thus $(\mathcal{N}',\mathcal{T}',\overline{\mathcal{V}}')$ is locally ell-equivalent to $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})$.

It is not hard to see that if $(\tilde{\mathcal{N}}',\tilde{\mathcal{T}}',\tilde{\overline{\mathcal{V}}}')\in\mathcal{F}_{\text{ell}}$ is another element locally ell-equivalent to $(\mathcal{N},\mathcal{T},\overline{\mathcal{V}})$ with equivariant maps $\tilde{h}_a:\tilde{\mathcal{N}}_{V_a}\to\mathcal{N}_{V_a}$ such that $h_{ab}=\tilde{h}_a\tilde{h}_b^{-1}$, then $(\tilde{\mathcal{N}}',\tilde{\mathcal{T}}',\tilde{\overline{\mathcal{V}}}')$ is globally ell-equivalent to $(\mathcal{N}',\mathcal{T}',\overline{\mathcal{V}}')$, nor that every global ell-equivalence class of elements of $\mathcal{F}_{\text{ell}}$ which are locally ell-equivalent to $\mathcal{N}$ determines a unique element of $H^1(\mathcal{B},\mathscr{I}^{\overline{\mathcal{V}}}(\mathcal{N})))$. ☐

5. The bundle $\overline{\mathcal{V}}$ and the orbits of the structure group G

In order to analyze the condition that an element $h\in\mathcal{I}^\infty(\mathcal{N}_V)$, $V\subset\mathcal{B}$ open, preserves $\overline{\mathcal{V}}$ we need to analyze the relation between the structure bundle $\overline{\mathcal{V}}$ and the orbits of G. Let β be some $\overline{\mathbb{D}}$-closed section of $\overline{\mathcal{V}}^*$ with $\iota_{\mathcal{T}}\beta=-i$ and let $\overline{\mathcal{K}}\subset\overline{\mathcal{V}}$ be its kernel. As in the proof of Theorem 4.2, let θ be the real one-form that vanishes on $\mathcal{K}\oplus\overline{\mathcal{K}}$ and satisfies $\langle\theta,\mathcal{T}\rangle=1$. The complexification of the kernel of θ, being the direct sum of subbundles $\mathcal{K}$ and $\overline{\mathcal{K}}$, determines an almost complex structure $J : \ker\theta \to \ker\theta$, $v + iJv \in \overline{\mathcal{K}}$ if $v \in \ker\theta$.

Let $g \in G$ and let $\{t_\nu\}$ be a sequence such that $\mathfrak{a}_{t_\nu} \to g$. If $v \in \overline{\mathcal{V}}_p$, then $d\mathfrak{a}_{t_\nu} v \in \overline{\mathcal{V}}_{\mathfrak{a}_{t_\nu} p}$, so $d\mathfrak{A}_g v \in \overline{\mathcal{V}}_{\mathfrak{A}_g p}$. Thus $\operatorname{Exp}(f)$ preserves $\overline{\mathcal{V}}$ whenever $f : \mathcal{N} \to \mathfrak{g}$ is constant.

The one-parameter subgroup $t \mapsto \mathfrak{a}_t$ of G gives an element $\hat{\mathcal{T}} \in \mathfrak{g}$. Let $\hat{Y}_1, \dots, \hat{Y}_d$ be a basis of $\mathfrak{g}$ with $\hat{Y}_d = \hat{\mathcal{T}}$ with dual basis $\hat{\theta}^1, \dots, \hat{\theta}^d$. Via the action of G the $\hat{Y}_j$ give smooth globally defined vector fields $Y_1, \dots, Y_d$ on $\mathcal{N}$, $Y_d = \mathcal{T}$,

$$Y_j(p) = \frac{d}{ds}\bigg|_{s=0} \operatorname{Exp}(s\hat{Y}_j)(p).$$

The vector fields Y_j are tangent to the orbits of G and pointwise linearly independent on $\mathcal{N}_{\mathrm{reg}}$ but not necessarily so on $\mathcal{N}_{\mathrm{sing}} = \mathcal{N}\backslash\mathcal{N}_{\mathrm{reg}}$. Clearly $[Y_j, Y_k] = 0$ for all j, k. Let

$$X_k = Y_k - \langle\theta, Y_k\rangle\mathcal{T}, \quad k = 1, \dots, d-1,$$

so that X_k is a section of $\ker\theta$. Then JX_k is defined and is another smooth section of $\ker\theta$. Since the functions $\langle\theta, Y_k\rangle$ are constant on orbits of G,

$$[Y_j, X_k] = 0, \quad j = 1, \dots, d,\ k = 1, \dots, d-1,$$
$$[X_j, X_k] = 0, \quad j, k = 1, \dots, d-1.$$

The vector fields $X_k + iJX_k$ are sections of $\overline{\mathcal{K}}$ and so are the sections

$$\operatorname{Exp}(s\hat{Y}_j)_*(X_k + iJX_k) = \operatorname{Exp}(s\hat{Y}_j)_* X_k + i\operatorname{Exp}(s\hat{Y}_j)_* JX_k.$$

since $\operatorname{Exp}(s\hat{Y}_j)$ preserves $\overline{\mathcal{V}}$ for each $s \in \mathbb{R}$. Thus $J\operatorname{Exp}(s\hat{Y}_j)_* X_k = \operatorname{Exp}(s\hat{Y}_j)_* JX_k$, so

$$\operatorname{Exp}(-s\hat{Y}_j)_* J \operatorname{Exp}(s\hat{Y}_j)_* X_k = JX_k.$$

It follows that $[Y_j, JX_k] = 0$ for all k, j, and thus

$$[X_j, JX_k] = [Y_j - \langle\theta, Y_j\rangle\mathcal{T}, JX_k] = JX_k\langle\theta, Y_j\rangle\mathcal{T}.$$

The brackets

$$[X_j + iJX_j, X_k + iJX_k] = -[JX_j, JX_k] + i([X_j, JX_k] + [JX_j, X_k])$$

are sections of $\overline{\mathcal{K}}$, in particular of $\ker\theta$, so $\langle\theta, [X_j, JX_k] + [JX_j, X_k]\rangle = 0$. But

$$[X_j, JX_k] + [JX_j, X_k] = (JX_k\langle\theta, Y_j\rangle - JX_j\langle\theta, Y_k\rangle)\mathcal{T},$$

hence $JX_k\langle\theta, Y_j\rangle - JX_j\langle\theta, Y_k\rangle = 0$, that is, $[X_j + iJX_j, X_k + iJX_k] = -[JX_j, JX_k]$. Therefore $[JX_j, JX_k] = 0$. Thus the vector fields

$$Y_1, \dots, Y_d, JX_1, \dots, JX_{d-1} \tag{5.1}$$

commute with each other.

For a real vector field W on $\mathcal{N}$ let $\exp(sW)$ denote the one-parameter group it determines. Define

$$F(t, s)(p) = \exp(t^1 Y_1) \circ \cdots \circ \exp(t^d Y_d) \circ \exp(s^1 JX_1) \circ \cdots \circ \exp(s^{d-1} JX_{d-1})(p)$$

for $(t,s) \in \mathbb{R}^d \times \mathbb{R}^{d-1}$. Then

$$\operatorname{Suss}(p) = \{F(t,s)(p) : (t,s) \in \mathbb{R}^d \times \mathbb{R}^{d-1}\}$$

is the Sussmann orbit through p of the family (5.1), see Sussmann [10]; these orbits are immersed submanifolds of $\mathcal{N}$.

For any $p \in \mathcal{N}$ the intersection $\overline{\mathcal{W}}_{\overline{\mathcal{O}}_p} = \overline{\mathcal{K}} \cap \mathbb{C}T\overline{\mathcal{O}}_p$ is a CR structure on $\overline{\mathcal{O}}_p$ (which could be just the 0 bundle), pointwise spanned by linear combinations $\sum_{k=1}^{d-1} b^k(X_k + iJX_k)$ such that $\sum_{k=1}^{d-1} b^k JX_k$ is again tangent to $\overline{\mathcal{O}}_p$:

$$\sum_{k=1}^{d-1} b^k JX_k = \sum_{j=1}^{d} a^j Y_j$$

with suitable a^j. Because the vector fields (5.1) commute, if this relation holds at p, then it holds at every point of $\operatorname{Suss}(p)$ with constant a^j, b^k. This gives $F(t,s)_*\overline{\mathcal{W}}_{\overline{\mathcal{O}}_p} = \overline{\mathcal{W}}_{\overline{\mathcal{O}}_{F(t,s)(p)}}$ for each s, t. Note in passing that $\mathcal{W}_{\overline{\mathcal{O}}_p} \oplus \overline{\mathcal{W}}_{\overline{\mathcal{O}}_p}$ is involutive, so

$$\overline{\mathcal{W}}_{\overline{\mathcal{O}}_p} = 0 \text{ if } \overline{\mathcal{K}} \text{ is nondegenerate at } p. \tag{5.2}$$

Let $d_p = \dim \overline{\mathcal{O}}_p$ and let n_p be the CR codimension of $\overline{\mathcal{W}}_{\overline{\mathcal{O}}_p}$. Then the dimension of the span of the vector fields (5.1) at p is $d_p + n_p$ (so $\dim \operatorname{Suss}(p) = d_p + n_p$). It also follows from the fact that the vector fields (5.1) commute that $\dim \overline{\mathcal{O}}_{F(t,s)(p)}$ is independent of t and s. Also $\overline{\mathcal{K}} \cap \operatorname{Suss}(p)$ is a CR structure, necessarily of CR codimension 1, spanned by the vector fields $X_j + iJX_j$.

Given any $p \in \mathcal{N}$ we may choose the basis $\hat{Y}_1, \dots, \hat{Y}_{n-1}, \hat{Y}_d$ of $\mathfrak{g}$ ($\hat{Y}_d = \hat{\mathcal{T}}$) so that with $m_p = \operatorname{rank} \overline{\mathcal{W}}_{\overline{\mathcal{O}}_p}$,

$$\begin{gathered} X_j = Y_j,\ j = 1, \dots, d,\ JX_j = X_{j+m_p},\ j = 1, \dots, m_p \\ \operatorname{span}\{JX_k : k = 2m_p + 1, \dots, 2m_p + n_p\} \cap T\overline{\mathcal{O}}_p = 0, \\ Y_k = 0,\ k = 2m_p + n_p + 1, \dots, d-1. \end{gathered} \tag{5.3}$$

holds along $\overline{\mathcal{O}}_p$.

Suppose now that $V \subset \mathcal{B}$ is open and $h = \operatorname{Exp}(f)$ with $f \in C^\infty(V, \mathfrak{g})$. Fix some $p_0 \in \mathcal{N}_V$. If $v \in \mathbb{C}T_{p_0}\mathcal{N}$, then, with f_{p_0} denoting the constant function $p \mapsto f(p_0)$

$$\begin{aligned} h_* v &= \operatorname{Exp}(f_{p_0})_* v + \sum_{j=1}^{d} v\langle \hat{\theta}^j, f\rangle Y_j \\ &= \operatorname{Exp}(f_{p_0})_* v + \sum_{j=1}^{d} \langle \theta, Y_j\rangle v\langle \hat{\theta}^j, f\rangle\, \mathcal{T} + \sum_{j=1}^{d-1} v\langle \hat{\theta}^j, f\rangle X_j. \end{aligned} \tag{5.4}$$

If $d = 1$, then

$$h_* v = \operatorname{Exp}(f_{p_0})_* v + v\langle \hat{\theta}^1, f\rangle \mathcal{T} \quad \text{for all } v \in \overline{\mathcal{V}}_{p_0},$$

so any $h \in \mathcal{I}^\infty(\mathcal{N}_V)$ preserves $\overline{\mathcal{V}}$. Assume then that $d > 1$. The vector $h_* v$ in (5.4) belongs to $\overline{\mathcal{V}}$ for every $v \in \overline{\mathcal{V}}_{p_0}$ if and only if

$$\frac{1}{2}\sum_{j=1}^{d-1} v\langle\hat{\theta}^j, f\rangle(X_j - iJX_j),$$

the component of $h_* v$ in $\mathcal{K}$ according to the decomposition $\mathcal{K}\oplus\overline{\mathcal{V}}$ of $\mathbb{C}T\mathcal{N}$, vanishes for every such v. Using (5.4) with $v = X_k$ gives

$$h_* X_k = \operatorname{Exp}(f_{p_0})_* X_k + \sum_{j=1}^{d}\langle\theta, Y_j\rangle X_k\langle\hat{\theta}^j, f\rangle\,\mathcal{T} + \sum_{j=1}^{d-1} X_k\langle\hat{\theta}^j, f\rangle X_j,$$

$$h_* JX_k = \operatorname{Exp}(f_{p_0})_* JX_k + \sum_{j=1}^{d}\langle\theta, Y_j\rangle JX_k\langle\hat{\theta}^j, f\rangle\,\mathcal{T} + \sum_{j=1}^{d-1} JX_k\langle\hat{\theta}^j, f\rangle X_j.$$

Since $X_k\langle\hat{\theta}^j, f\rangle = 0$ for all j and $[Y_\ell, X_k] = 0$ for all ℓ, the first formula reduces to $h_* X_k = X_k$. If h preserves $\overline{\mathcal{V}}$, this gives $JX_k = h_* \operatorname{Exp}(h) JX_k$, so we must have $\sum_{j=1}^{d}\langle\theta, Y_j\rangle JX_k\langle\hat{\theta}^j, f\rangle = 0$ and $JX_k\langle\hat{\theta}^j, f\rangle = 0$ for $j = 1, \dots, d-1$, so $\langle\hat{\theta}^j, f\rangle$ is constant on the Sussmann orbits of the family (5.1).

Let $Z_j = \frac{1}{2}(X_j - iJX_j)$. Assuming that the basis $\hat{Y}_j$ satisfies (5.3) along $\overline{\mathcal{O}}_{p_0}$, the condition that h preserves $\overline{\mathcal{V}}$ is that

$$\sum_{j=1}^{m_{p_0}} v(\langle\hat{\theta}^j, f\rangle + i\langle\hat{\theta}^{j+m_{p_0}}, f\rangle)Z_j + \sum_{j=2m_{p_0}+1}^{2m_{p_0}+d_{p_0}} v\langle\hat{\theta}^j, f\rangle\, Z_j = 0$$

along $\overline{\mathcal{O}}_{p_0}$. This gives

$$\overline{\mathbb{D}}\langle\hat{\theta}^j, f\rangle = -\overline{\mathbb{D}}\langle\hat{\theta}^{j+m_{p_0}}, f\rangle,\ j = 1, \dots, m_{p_0},\ \overline{\mathbb{D}}\langle\hat{\theta}^j, f\rangle = 0,\ j = 2m_{p_0} + 1, \dots n_{p_0}$$

at p_0.

If this is the case, then

$$h_* v = \operatorname{Exp}(f_{p_0})_* v + \sum_{j=1}^{d}\langle\hat{\theta}, Y_j\rangle v\langle\hat{\theta}^j, f\rangle\,\mathcal{T} + \frac{1}{2}\sum_{j=1}^{d-1} v\langle\hat{\theta}^j, f\rangle(X_j + iJX_j)$$

whenever $v \in \overline{\mathcal{V}}_{p_0}$.

If $\overline{\mathcal{W}}_{\overline{\mathcal{O}}_{p_0}} = 0$ (for instance if $\overline{\mathcal{K}}_{p_0}$ is nondegenerate, see (5.2)), then the condition that $h = \operatorname{Exp}(f)$ preserves $\overline{\mathcal{V}}$ is that $\overline{\mathbb{D}}\langle\hat{\theta}^j, f\rangle = 0$, $j = 1, \dots, n_{p_0}$. So if $\overline{\mathcal{K}}$ is nondegenerate on a dense subset of $\mathcal{N}$, then $\overline{\mathbb{D}}\langle\hat{\theta}^j, f\rangle = 0$, $j = 1, \dots, d-1$ in $\mathcal{N}_V$. Since the functions $\langle\hat{\theta}^j, f\rangle$ are real valued, they must be locally constant.

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Solvability of Planar Complex Vector Fields with Applications to Deformation of Surfaces

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Dedicated to Linda P. Rothschild

Abstract. Properties of solutions of a class of semilinear equations of the form $Lu = f(x, y, u)$, where L is a $\mathbb{C}$-valued planar vector field is studied. Series and integral representations are obtained in a tubular neighborhood of the characteristic curve of L. An application to infinitesimal bendings of surfaces with nonnegative curvature in $\mathbb{R}^3$ is given.

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Introduction

This paper deals with the properties of the solutions of semiglobal equations related to complex-valued vector fields in the plane as well as their applications to bending of surfaces in $\mathbb{R}^3$.

The typical vector field L considered here is smooth in an open set $\mathcal{O} \subset \mathbb{R}^2$ and such that L and $\overline{L}$ are independent except along a simple closed curve Σ. The vector field L is of infinite type along Σ and $L \wedge \overline{L}$ vanishes to first order on Σ (see Section 2 for details). Properties of the solutions of the equation $Lu = f(x, y, u)$ are sought in a full neighborhood of the characteristic set Σ. Such equations are then used to model infinitesimal bendings of surfaces with nonnegative curvature. This work is motivated by results contained in the papers [1], [2], [3], [4], [5], [6], [7], [9], [10], [21], [22].

Our approach to the study of the equation $Lu = f$ in a tubular neighborhood of Σ is to reduce it into the study of a CR equation with a punctual singularity. Namely to an equation of the form $\overline{z}w_{\overline{z}} = f$ in a neighborhood of $0 \in \mathbb{C}$. For this reason we start, in Section 1, by listing recent results about the behavior of the solutions of the CR equation. In Section 2, we recall the normal forms for the vector field L. The simple normal form allows the transition to the singular CR equation. In Section 3, the solutions in a neighborhood U of Σ, of the $\mathbb{R}$-linear equation $Lu = Au + B\overline{u}$ are characterized. A uniqueness property is then deduced. That is, we show that if u is a continuous solution that vanishes on a sequence of points p_k contained in a connected component U_1 of $U\backslash\Sigma$, and if p_k accumulates on Σ, then $u \equiv 0$ on U_1. In Section 4, Schauder's Fixed Point Theorem is used to study a semilinear equation. The last two sections deal with bendings of surfaces. In Section 5, we reduce the equations for the bending fields of a surface with nonnegative curvature into solvability of (complex) vector field of asymptotic directions. In Section 6, we show the nonrigidity of surfaces with positive curvature except at a flat point. In particular, if S is a germ of such a surface, then for every $\epsilon > 0$, there exist surfaces S_1 and S_2 that are ϵ-close from S, in the C^k-topology, and such that S_1 and S_2 are isometric but not congruent.

1. A singular CR equation

We describe here recent results of the author [18] about the properties of solutions of a CR equation with a punctual singularity (see also [12] [13] and [23]). These properties will be used to study more general equations associated with complex vector fields.

Let $a(\theta)$ and $b(\theta)$ be 2π-periodic, $\mathbb{C}$-valued functions of class C^k with $k \geq 2$. Consider the equation

$$\frac{\partial w}{\partial \overline{z}} = \frac{a(\theta)}{2\overline{z}} w + \frac{b(\theta)}{2\overline{z}} \overline{w}\,, \tag{1.1}$$

where $z = re^{i\theta}$. We will assume throughout that the average of the function a is real:

$$\frac{1}{2\pi} \int_0^{2\pi} a(\theta) d\theta \,\in\, \mathbb{R}\,. \tag{1.2}$$

By using periodic ODEs, we construct basic solutions to (1.1).

Proposition 1.1. ([18]) *There exists a sequence of real numbers*

$$\cdots < \lambda_{-1}^- \leq \lambda_{-1}^+ < \lambda_0^- \leq \lambda_0^+ < \lambda_1^- \leq \lambda_1^+ < \cdots$$

with $\lambda_j^{\pm} \longrightarrow \pm\infty$ *as* $j \longrightarrow \pm\infty$ *and there exists a sequence of functions* $\psi_j^{\pm}(\theta) \in C^{k+1}(\mathbb{S}^1, \mathbb{C})$ *such that* $w_j^{\pm}(r,\theta) = r^{\lambda_j^{\pm}} \psi_j^{\pm}(\theta)$ *is a solution of* (1.1) *in* $\mathbb{R}^+ \times \mathbb{S}^1$. *Furthermore, for each* $j \in \mathbb{Z}$, *the rotation of* $\psi_j^{\pm}$ *is* $2\pi j$.

Remark 1.1 The $\lambda_j^{\pm}$'s and the $\psi_j^{\pm}$'s are the eigenvalues and eigenfunctions of the first-order ODE

$$i\psi'(\theta) = (a(\theta) - \lambda)\psi(\theta) + b(\theta)\overline{\psi(\theta)}.$$

Note that this differential equation has (locally) two $\mathbb{R}$-independent solutions. $\lambda_{j_0}^{\pm}$ are then the eigenvalues corresponding to periodic solutions with rotation $2\pi j_0$. If the ODE has two independent periodic solution, then $\lambda_{j_0}^- = \lambda_{j_0}^+$ otherwise $\lambda_{j_0}^- < \lambda_{j_0}^+$.

These basic solutions are then used to construct kernels from which a generalized Cauchy formula is derived. More precisely, for $\zeta = \rho e^{i\alpha}$ and $z = re^{i\theta}$, define $\Omega_1(z,\zeta)$ and $\Omega_2(z,\zeta)$ by

$$\begin{aligned}
\Omega_1(z,\zeta) &= \begin{cases} \dfrac{-1}{2}\displaystyle\sum_{\lambda_j^\pm \geq 0}\left(\frac{r}{\rho}\right)^{\lambda_j^\pm}\psi_j^\pm(\theta)\overline{\psi_j^\pm(\alpha)} & \text{if } r<\rho \\ \dfrac{1}{2}\displaystyle\sum_{\lambda_j^\pm < 0}\left(\frac{r}{\rho}\right)^{\lambda_j^\pm}\psi_j^\pm(\theta)\overline{\psi_j^\pm(\alpha)} & \text{if } \rho<r \end{cases} \\
\Omega_2(z,\zeta) &= \begin{cases} \dfrac{-1}{2}\displaystyle\sum_{\lambda_j^\pm \geq 0}\left(\frac{r}{\rho}\right)^{\lambda_j^\pm}\psi_j^\pm(\theta)\psi_j^\pm(\alpha) & \text{if } r<\rho \\ \dfrac{1}{2}\displaystyle\sum_{\lambda_j^\pm < 0}\left(\frac{r}{\rho}\right)^{\lambda_j^\pm}\psi_j^\pm(\theta)\psi_j^\pm(\alpha) & \text{if } \rho<r \end{cases}
\end{aligned} \tag{1.3}$$

By studying the asymptotic behavior of the spectral values $\lambda_j^\pm$ and of $\psi_j^\pm$ as $j \longrightarrow \pm\infty$, we can estimate the singularities of Ω_1 and Ω_2. It is found (see [18]), that the singularity of Ω_1 is of the form $\left|\dfrac{z}{\zeta}\right|^\nu \dfrac{\zeta}{z-\zeta}$ and that of Ω_2 is of the form $\left|\dfrac{z}{\zeta}\right|^\nu \log\dfrac{\zeta}{\zeta - z}$, where $\nu \in [0,\ 1)$ is the fractional part of the average of the coefficient $a(\theta)$.

Theorem 1.1. ([18]) *Let $U \subset \mathbb{C}$ be an open and bounded set whose boundary ∂U consists of finitely many simple closed curves. If $w \in C^0(\overline{U})$ solves* (1.1), *then for $z \in U$, we have*

$$w(z) = \frac{-1}{2\pi i}\int_{\partial U}\left[\frac{\Omega_1(z,\zeta)}{\zeta}w(\zeta)d\zeta - \frac{\Omega_2(z,\zeta)}{\overline{\zeta}}\overline{w(\zeta)}d\overline{\zeta}\right]. \tag{1.4}$$

Remark 1.2 The spectral values $\lambda_j^\pm$ give the possible orders at the singularity 0 of the zeros (if $\lambda_j^\pm > 0$) or poles (if $\lambda_j^\pm < 0$) the solutions of (1.1). In the generic case, $\lambda = 0$ is not a spectral value. When $\lambda = 0$ is a spectral value, it gives rise to a solution $w(r,\theta) = \psi(\theta)$ of (1.1). This solution is bounded but not continuous, unless it is constant. This latter case occurs only in the special case when the coefficients $a(\theta)$ and $b(\theta)$ satisfy $b(\theta) = a(\theta)e^{i\theta_0}$ for some constant θ_0.

Remark 1.3 The solutions of (1.1) admit series representations. More precisely, if u is a continuous solution of (1.1) in the disc $D(0,R)$, then

$$u(r,\theta) = \sum_{\lambda_j^\pm \geq 0}\left(C_j^-\psi_j^-(\theta) + C_j^+\psi_j^+(\theta)\right)$$

where $C_j^\pm \in \mathbb{R}$.

For the nonhomogeneous equation we have the following representation.

Theorem 1.2. ([18]) *Let U be as in Theorem 1.1, $F \in L^p(\overline{U})$ with $p > 2/(1-\nu)$ and suppose $w \in C^0(\overline{U})$ satisfies the equation*

$$\frac{\partial w}{\partial \overline{z}} = \frac{a(\theta)}{2\overline{z}} w + \frac{b(\theta)}{2\overline{z}} \overline{w} + F(z)\,. \tag{1.5}$$

Then w has the representation

$$\begin{aligned} w(z) = \frac{-1}{2\pi i} \int_{\partial U} \left[\frac{\Omega_1(z,\zeta)}{\zeta} w(\zeta) d\zeta - \frac{\Omega_2(z,\zeta)}{\overline{\zeta}} \overline{w(\zeta) d\zeta} \right] + \\ + \frac{1}{\pi} \iint_U \left(\frac{\Omega_1(z,\zeta)}{\zeta} F(\zeta) + \frac{\Omega_2(z,\zeta)}{\overline{\zeta}} \overline{F(\zeta)} \right) d\xi d\eta. \end{aligned} \tag{1.6}$$

The second integral appearing in (1.6) is a particular solution of (1.5). It defines an integral operator

$$T_R F(z) = \frac{1}{\pi} \iint_U \left(\frac{\Omega_1(z,\zeta)}{\zeta} F(\zeta) + \frac{\Omega_2(z,\zeta)}{\overline{\zeta}} \overline{F(\zeta)} \right) d\xi d\eta \tag{1.7}$$

It is shown in [18] that $T_R : L^p(\overline{U}) \longrightarrow C^0(\overline{U})$ is a compact operator if 0 is not a spectral value. If 0 is a spectral value a slight modification of Ω_1 and Ω_2 (removing in the sums (1.3) the contribution of the zero eigenvalue) gives a compact operator. Furthermore, there is $\delta > 0$ so that T_R satisfies

$$|T_R F(z)| \le C R^{\delta} ||F||_p, \quad \forall z \in D(0,R), \ \forall f \in L^p(D(0,R)). \tag{1.8}$$

2. Normal form of a class of vector fields

We reduce a complex vector field satisfying certain properties into a simple canonical form. Let

$$L = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$

be a vector field with coefficients $a, b \in C^\infty(\mathcal{O}, \mathbb{C})$, where $\mathcal{O} \subset \mathbb{R}^2$ is open. We assume that L satisfies the following conditions

(i) L is elliptic everywhere in $\mathcal{O}$ except along a smooth simple closed curve $\Sigma \subset \mathcal{O}$. Thus, L and $\overline{L}$ are independent on $\mathcal{O}\backslash\Sigma$ and dependent on Σ;

(ii) L is of infinite type on Σ. That is, L, $\overline{L}$, and any Lie bracket $[X_1, [X_2, [\cdots [X_{k-1}, X_k] \cdots]]]$ are dependent on Σ, where each X_i is either L or $\overline{L}$;

(iii) The bivector $L \wedge \overline{L}$ vanishes to first order along Σ.

The first condition means that for $p \in \mathcal{O}\backslash\Sigma$, we can find local coordinates (x, y), centered at p, in which L is a multiple of the CR operator $\dfrac{\partial}{\partial \overline{z}}$. The second condition means that for $p \in \Sigma$, there are local coordinates (x, y) around p, in which Σ is given by $x = 0$ and L is a multiple of a vector field $\dfrac{\partial}{\partial y} + ic(x,y)\dfrac{\partial}{\partial x}$ where the function $c(x, y)$ satisfies $c(0, y) = 0$. The third condition means that c vanishes to first order ($c(x,y) = x\alpha(x,y)$ with $\alpha(0,0) \neq 0$). The following theorem contained in [11] gives the canonical form for such vector fields.

Theorem 2.1. ([11]) *Assume that L is as above and satisfies conditions* (i), (ii), *and* (iii). *Then, there exist $c \in \mathbb{R}^* + iR$, open sets $U \subset \mathcal{O}$, $V \subset \mathbb{R} \times \mathbb{S}^1$, with $\Sigma \subset U$, $\{0\} \times \mathbb{S}^1 \subset V$, and a diffeomorphism $\Phi : U \longrightarrow V$ such that*

$$\Phi_* L = m(r,\theta)\left(ic\frac{\partial}{\partial\theta} + r\frac{\partial}{\partial r}\right),$$

where (r,θ) are the coordinates in $\mathbb{R} \times \mathbb{S}^1$ and m is a nonvanishing function.

The number c is an invariant of the structure defined by L in $\mathcal{O}$. More precisely, if $\omega = bdx - ady$ (the dual form of L), then $d\omega = \omega \wedge \alpha$ for some 1-form α and

$$\frac{1}{c} = \frac{1}{2\pi i}\int_\Sigma \alpha.$$

Without loss of generality, we can assume that $Re(c) > 0$ (otherwise replacing the coordinate θ by $-\theta$ will transform it to a vector field with the desired c). If $\mathrm{Im}c \neq 0$, then for every $k \in \mathbb{Z}^+$, the diffeomorphism Φ in the theorem can be taken to be of class C^k (see [11] for details). When $\mathrm{Im}c = 0$, then Φ is, in general, only of class C^1. However, it is proved in [8], that if $c \in \mathbb{R}\backslash\mathbb{Q}$, then again, for every $k \in \mathbb{Z}^+$, the diffeomorphism Φ can be taken in C^k. Note also that normal forms for vector fields with higher order of vanishing along Σ are obtained in [15].

It follows from Theorem 2.1 that the study of the equation

$$Lu = g(u,x,y)$$

in a tubular neighborhood of the characteristic set Σ is reduced to the study of the equation

$$Xu = f(u,r,\theta)$$

in a neighborhood of the circle $\Sigma_0 = \{0\} \times \mathbb{S}^1$ in $\mathbb{R} \times \mathbb{S}^1$, where X is the vector field

$$X = ic\frac{\partial}{\partial\theta} + r\frac{\partial}{\partial r}\,. \tag{2.1}$$

Throughout the remainder, and for simplicity, we will assume that $c \in \mathbb{R}^+$.

3. Solvability for the vector field X

We use the singular CR equation (1.1) to describe the solutions for X in a neighborhood of the characteristic set. Let $a_0(\theta),\ b_0(\theta) \in C^k(\mathbb{S}^1, \mathbb{C})$ with $k \geq 2$ and such that the average of a_0 is real. Consider the equation

$$Xu = (a_0(\theta) + ra_1(r,\theta))u + (b_0(\theta) + rb_1(r,\theta))\overline{u}\,, \tag{3.1}$$

where a_1 and b_1 are continuous functions defined in $A_R = (-R,\ R) \times \mathbb{S}^1$. We will use the notation

$$A_R^+ = A_R \cap \mathbb{R}^+ \times \mathbb{S}^1 \quad \text{and} \quad A_R^- = A_R \cap \mathbb{R}^- \times \mathbb{S}^1\,.$$

With (3.1), we associate the CR equation

$$\frac{\partial w}{\partial \overline{z}} = \frac{a_0(\theta)}{2c\overline{z}} w + \frac{b_0(\theta)}{2c\overline{z}} \overline{w} . \tag{3.2}$$

Let $r^{\lambda_j^\pm} \psi_j^\pm(\theta)$ be the basic solutions of (3.2). We assume that 0 is not a spectral value of (3.2). We have the following representation of the solutions of (3.1)

Theorem 3.1. *Let $u(r,\theta)$ be a bounded solution of* (3.1) *in $\overline{A_R^+}$ (resp. $\overline{A_R^-}$). Then there exists a spectral value $\lambda_j^\pm > 0$ such that*

$$u(r,\theta) = r^{c\lambda_j^\pm} \psi_j^\pm(\theta) P(r,\theta) , \tag{3.3}$$

where P is a continuous function in $\overline{A_R^+}$ (resp. $\overline{A_R^-}$) with $P(0,\theta) \neq 0$ for every θ. Conversely, for every positive spectral value $\lambda_j^\pm$, there exists P as above so that u given by (3.3) *solves* (3.1).

Proof. The function $z = |r|^c e^{i\theta}$ satisfies $Xz = 0$. Furthermore,

$$\Phi^\pm : R^\pm \times \mathbb{S}^1 \longrightarrow \mathbb{C}\backslash 0; \quad \Phi^\pm(r,\theta) = z = |r|^c e^{i\theta}$$

is a diffeomorphism. Since $X\overline{z} = 2c\overline{z}$, then the pushforward of equation (3.1) via Φ^+ in A_R^+ into the punctured disc $D(0,R^c)\backslash 0$ in $\mathbb{C}$ is the CR equation

$$w_{\overline{z}} = \left(\frac{a_0}{2c\overline{z}} + |z|^{c'} a_1(z) \right) w + \left(\frac{b_0}{2c\overline{z}} + |z|^{c'} b_1(z) \right) \overline{w} , \tag{3.4}$$

where a_1 and b_1 are bounded functions and where the exponent c' is $c' = \dfrac{1}{c} - 1$. It is shown in [13] that the solutions of equations (3.4) are similar to those of (3.2). Thus each solution w of (3.4) can be written as $w = w_0 P$ for some solution w_0 of (3.2) and some non vanishing continuous function P and vice versa. Now w_0 is similar to a basic solution (see Remark 1.3) and consequently, if u solves (3.1) in $\overline{A_R^+}$ (or in $\overline{A_R^-}$), then $w = u \circ \Phi^+$ (or $w = u \circ \Phi^-$) solves (3.4) and the conclusion of the theorem follows. □

Remark 3.1 If u is a bounded solution of (3.1) in A_R, then necessarily u vanishes on the circle Σ_0 but the orders of vanishing from $r > 0$ and from $r < 0$ might be different.

As a consequence of Theorem 3.1, we have the following uniqueness result.

Theorem 3.2. *Let u be a bounded solution of* (3.1) *in A_R^+ (resp. A_R^-). If u vanishes on a sequence of points $\{p_k\}_k \subset A_R^+$ (resp. A_R^-) such that p_k converges to a points $p_0 \in \Sigma_0$ then $u \equiv 0$*

Proof. It follows from Theorem 3.1, that it is enough to prove the uniqueness for solutions of the model equation (3.2) that vanish on a sequence of points z_k that converges to $0 \in \mathbb{C}$. Let then w be such a solution of (3.2). We use the series expansion of w (see Remark 1.3) to show that $w \equiv 0$. By contradiction, suppose that $w \not\equiv 0$, then there exists $\lambda_{j_0}^{\pm} > 0$ and $d_{j_0}^{\pm} \in \mathbb{R}$ with $d_{j_0}^{-} \neq 0$ or $d_{j_0}^{+} \neq 0$ such that

$$w(z) = d_{j_0}^{-} r^{\lambda_{j_0}^{-}} \psi_{j_0}^{-}(\theta) + d_{j_0}^{+} r^{\lambda_{j_0}^{+}} \psi_{j_0}^{+}(\theta) + \sum_{j>j_0} (d_j^{-} r^{\lambda_j^{-}} \psi_j^{-}(\theta) + d_j^{+} r^{\lambda_j^{+}} \psi_j^{+}(\theta)) \quad (3.5)$$

Set $z_k = r_k e^{i\theta_k}$ with $r_k \to 0$ and $\theta_k \to \theta_0$ (otherwise take a subsequence of θ_k). After replacing z by z_k in (3.5) and dividing by $r_k^{\lambda_{j_0}^{-}}$, we get

$$0 = d_{j_0}^{-} \psi_{j_0}^{-}(\theta_k) + d_{j_0}^{+} r_k^{\lambda_{j_0}^{+} - \lambda_{j_0}^{-}} \psi_{j_0}^{+}(\theta_k) + o(r_k^{\lambda_{j_0}^{+} - \lambda_{j_0}^{-}})$$

Now, we let $k \to \infty$ to obtain

$$\begin{array}{ll} d_{j_0}^{-} \psi_{j_0}^{-}(\theta_0) = 0 & \text{if } \lambda_{j_0}^{+} > \lambda_{j_0}^{-}; \text{ and} \\ d_{j_0}^{-} \psi_{j_0}^{-}(\theta_0) + d_{j_0}^{+} \psi_{j_0}^{+}(\theta_0) = 0 & \text{if } \lambda_{j_0}^{+} = \lambda_{j_0}^{-}. \end{array}$$

In the first case, we get $d_{j_0}^{-} = 0$ since $\psi_{j_0}^{-}$ is nowhere 0 (as a non trivial solution of a first-order linear ODE). By repeating the argument we also find $d_{j_0}^{+} = 0$. In the second case, we get $d_{j_0}^{-} = d_{j_0}^{+} = 0$ since, $\psi_{j_0}^{-}$ and $\psi_{j_0}^{+}$ are independent solutions of the same linear homogeneous ODE (see Remark 1.1). In both cases, we get a contradiction and so $w \equiv 0$. □

4. A semilinear equation

In this section we use Schauder's Fixed Point Theorem and the operator T of section 1 to construct non trivial solutions of a semilinear equation for the vector field X. Let $f(u, r, \theta)$ be a function of class C^k, with $k \geq 2$, defined in a neighborhood of $\{0\} \times \{0\} \times \mathbb{S}^1$ in $\mathbb{C} \times \mathbb{R} \times \mathbb{S}^1$ and such that

$$f(0, r, \theta) = 0 \quad \text{and} \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial f}{\partial u}(0, 0, \theta) d\theta \in \mathbb{R}. \quad (4.1)$$

Set

$$a_0(r, \theta) = \frac{\partial f}{\partial u}(0, r, \theta) \quad \text{and} \quad b_0(r, \theta) = \frac{\partial f}{\partial \overline{u}}(0, r, \theta). \quad (4.2)$$

With the equation

$$Xu = f(u, r, \theta) \quad (4.3)$$

we associate the CR equation

$$\frac{\partial w}{\partial \overline{z}} = \frac{a_0(\theta)}{2c\overline{z}} w + \frac{b_0(\theta)}{2c\overline{z}} \overline{w} \quad (4.4)$$

We have the following theorem.

Theorem 4.1. *Let f, a_0, and b_0 be as above. Then there exists $\epsilon > 0$ such that for every basic solution $|z|^{\lambda_j^{\pm}} \psi_j^{\pm}(\theta)$ of (4.4), with $\lambda_j^{\pm} > 0$, there exists a nonvanishing function P defined in $\overline{A_\epsilon}$ such that*

$$u(r, \theta) = |r|^{c\lambda_j^{\pm}} \psi_j^{\pm}(\theta) P(r, \theta) \quad (4.5)$$

solves equation (4.3).

Proof. We write f as

$$f(u,r,\theta) = (a_0(\theta) + ra_1(r,\theta))u + (b_0(\theta) + rb_1(r,\theta))\overline{u} + g(u,r,\theta)$$

with $g = o(|u|^2)$ as $u \to 0$. We seek a solution of (4.3) in the form

$$u(r,\theta) = |r|^{c\lambda_j^\pm}\psi_j^\pm(\theta)\,(1 + v(r,\theta))$$

with $v(0,\theta) = 0$. The function v must then satisfy the equation

$$Xv = b_0(\theta)\frac{\overline{\psi_j^\pm(\theta)}}{\psi_j^\pm(\theta)}(\overline{v} - v) + ra_1(1+v) + rb_1\frac{\overline{\psi_j^\pm(\theta)}}{\psi_j^\pm(\theta)}(1+\overline{v}) + h(v,r,\theta) \tag{4.6}$$

with

$$h(v,r,\theta) = \frac{g(w(1+v),r,\theta)}{w}, \quad \text{and} \quad w = |r|^{c\lambda_j^\pm}\psi_j^\pm(\theta).$$

Since $g = o(|u|^2)$, we can write h as

$$h(v,r,\theta) = |r|^{c\lambda_j^\pm}k(v,r,\theta)$$

with k continuous and $k = 0$ for $v = 0$. The pushforward of equation (4.6), in say A_R^+, gives a singular semilinear CR equation of the form

$$\frac{\partial w}{\partial \overline{z}} = \frac{B_0(\theta)}{2c\overline{z}}(\overline{w} - w) + |z|^{1/c-1}A(z) + |z|^{\lambda_j^\pm/c-1}F(w,z) \tag{4.7}$$

where $B_0 = b_0\dfrac{\overline{\psi_j^\pm}}{\psi_j^\pm}$ and $A(z)$ is a bounded function near $0 \in \mathbb{C}$ and where F is a bounded function, continuous in the variable w, and $F(0,z) = 0$. We are going to prove that (4.7) has a continuous solution w with $w(0) = 0$.

First we need to verify that the average of B_0 is a real number. Since the function $\psi_j^\pm$ satisfies the ODE

$$i\frac{d\psi_j^\pm}{d\theta} = (a_0(\theta) - \lambda_j^\pm)\psi_j^\pm + b_0(\theta)\overline{\psi_j^\pm},$$

then

$$\int_0^{2\pi} B_0 d\theta = \int_0^{2\pi} b_0\frac{\overline{\psi_j^\pm}}{\psi_j^\pm}d\theta = i\int_0^{2\pi}\frac{d\psi_j^\pm}{\psi_j^\pm} - \int_0^{2\pi}(a_0 - \lambda_j^\pm)d\theta$$

is a real number because the average of a_0 is real.

For each function $s(z) \in C^0(D(0,R))$, consider the linear equation

$$\frac{\partial w}{\partial \overline{z}} = \frac{B_0(\theta)}{2c\overline{z}}(\overline{w} - w) + G(s(z),z) \tag{4.8}$$

with

$$G(s(z),z) = |z|^{1/c-1}A(z) + |z|^{\lambda_j^\pm/c-1}F(s(z),z).$$

We know from Section 1, that the function

$$T_R G(z) = \frac{1}{\pi}\iint_{D(0,R)}\left(\frac{\Omega_1(z,\zeta)}{\zeta}G(s(\zeta),\zeta) + \frac{\Omega_2(z,\zeta)}{\overline{\zeta}}\overline{G(s(\zeta),\zeta)}\right)d\xi d\eta$$

is a continuous solution of (4.8), satisfies $T_R G(0) = 0$ and $|T_R G(z)| \leq CR^\delta ||G||_p$, for some positive number δ. Consider the set

$$\Lambda_K = \{s \in C^0(D(0,R));\ s(0) = 0 \text{ and } ||s||_0 \leq K\}.$$

With the function G as defined above, there is a constant $\hat{K}$ (depending on K) such that $||G(s(z), z)||_p \leq \hat{K}$ for every $s \in \Lambda_K$. Thus $|T_R G(z)| \leq \hat{K} R^\delta$ for every $s \in \Lambda_K$. In particular, $T_R G \in \Lambda_K$ if R is small enough. Since Λ_K is convex and compact (Ascoli-Arzelà Theorem), then by Schauder's Fixed Point Theorem the operator T_R has a fixed point w which is a solution of (4.7). Consequently, by using the pullbacks, we obtain desired solutions to the semilinear equation (4.3). □

5. Equations for the bending fields

In the remainder of this paper, we consider how complex vector fields can be used to study deformations of surfaces in $\mathbb{R}^3$ with nonnegative curvature.

Let $S \subset \mathbb{R}^3$ be a C^∞ surface. Suppose that S is given by parametric equations:

$$S = \{R(s,t) = (x(s,t), y(s,t), z(s,t)) \in \mathbb{R}^3;\ (s,t) \in \mathcal{O}\},$$

where x, y, and z are C^∞ functions defined in a domain $\mathcal{O} \subset \mathbb{R}^2$. An infinitesimal bending of S of order n, of class C^k, is a deformation surface $S_\epsilon \subset \mathbb{R}^3$, with $\epsilon \in \mathbb{R}$ a parameter, and given by a position vector

$$R_\epsilon(x,y) = R(x,y) + 2\epsilon U^1(x,y) + 2\epsilon^2 U^2(x,y) + \cdots + 2\epsilon^n U^n(x,y)$$

such that each bending field $U^j : \mathcal{O} \longrightarrow \mathbb{R}^3$ is of class C^k and such that the first fundamental form of S_ϵ satisfies

$$dR_\epsilon^2 = dR^2 + o(\epsilon^n) \quad \text{as} \quad \epsilon \to 0. \tag{5.1}$$

Thus S_ϵ is an isometric approximation of S. The trivial bendings of S are those generated by the rigid motions of $\mathbb{R}^3$. In particular, for these motions, we have $U^1 = C \times R(s,t) + D$, where C, D, are constants in $\mathbb{R}^3$, and where $\times$ denotes the vector product in $\mathbb{R}^3$. It should be mentioned right away that the existence and characterization of nontrivial infinitesimal bendings for surfaces whose curvatures change signs is not clear even in the local setting. We refer to [19] and the references therein for an overview of problems related to these bendings. Some recent work of the author about bendings of surfaces can be found in [14], [16], [17]. Note that since,

$$dR_\epsilon^2 = dR^2 + 4\epsilon dR \cdot dU^1 + 4\sum_{j=1}^{n} \epsilon^j \left(dR \cdot dU^j + \sum_{k=2}^{j-1} dU^k \cdot dU^{j-k} \right) + o(\epsilon^n),$$

then condition (5.1) is equivalent to the system

$$\begin{aligned} dR\cdot dU^1 &= 0\\ dR\cdot dU^j &= -\sum_{k=1}^{j-1} dU^k\cdot dU^{j-k}, \quad j=2,\dots,n. \end{aligned} \tag{5.2}$$

Let $E,\ F,\ G$, and $e,\ f,\ g$ be the coefficients of the first and second fundamental forms of S. The Gaussian curvature of S is therefore

$$K=\frac{eg-f^2}{EG-F^2}\,.$$

The type of the system (5.2) depends on the sign of K. It is elliptic when $K>0$; hyperbolic when $K<0$; and parabolic when $K=0$. From now on, we will assume that S has nonnegative curvature ($K\geq 0$). Let L be the (complex) vector field of asymptotic directions of S. That is,

$$L=\frac{\partial}{\partial s}+\sigma(s,t)\frac{\partial}{\partial t}, \tag{5.3}$$

where

$$\sigma=-\frac{f+i\sqrt{eg-f^2}}{g} \tag{5.4}$$

is an asymptotic direction of S. Note that L is nondegenerate in the regions where $K>0$. With each bending field U^j, we associate the complex-valued function

$$w^j=LR\cdot U^j=m^j+\sigma n^j, \tag{5.5}$$

with $m^j=R_s\cdot U^j$ and $n^j=R_t\cdot U^j$. The following proposition reformulates the solvability of the U^j in terms of the solvability for the w^j's.

Proposition 5.1. *If $U^1,\dots,U^n$ solve* (5.2) *then the functions $w^1,\dots,w^n$ solve the system*

$$\begin{aligned} Lw^1 &= Aw^1+B\overline{w^1}\\ Lw^j &= Aw^j+B\overline{w^j}-\sum_{k=1}^{j-1} LU^k\cdot LU^{j-k}, \qquad j=2,\dots,n. \end{aligned} \tag{5.6}$$

where

$$A=\frac{(L^2R\times\overline{L}R)\cdot(LR\times\overline{L}R)}{(LR\times\overline{L}R)\cdot(LR\times\overline{L}R)}\,,\quad B=\frac{-(L^2R\times LR)\cdot(LR\times\overline{L}R)}{(LR\times\overline{L}R)\cdot(LR\times\overline{L}R)}\,.$$

Proof. A version of this proposition for the case $n=1$ is contained in [16]. Thus we need only to verify the nonhomogeneous equations for $j\geq 2$. Calculations similar to those used in [16] lead to the following system

$$V_s^j+QV_t^j+MV=K^j \tag{5.7}$$

where $V^j = \begin{pmatrix} m^j \\ n^j \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -e/g \\ 1 & -2f/g \end{pmatrix}$, $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$,

$$M_{11} = \frac{1}{g|R_s \times R_t|} R_t \cdot (R_{ss} + R_{tt}), \quad M_{12} = -\frac{1}{g|R_s \times R_t|} R_s \cdot (R_{ss} + R_{tt})$$
$$M_{21} = \frac{2}{g(e+g)|R_s \times R_t|} R_t \left[f(R_{ss} + R_{tt}) + gR_{st} \times (R_{ss} + R_{tt}) \right]$$
$$M_{22} = \frac{2}{g(e+g)|R_s \times R_t|} R_s \left[-f(R_{ss} + R_{tt}) + gR_{st} \times (R_{ss} + R_{tt}) \right]$$

and where

$$K^j = \begin{pmatrix} \alpha^j - \gamma^j e/g \\ \beta^j - 2\gamma^j f/g \end{pmatrix} \quad \text{with} \quad \begin{cases} \alpha^j &= -\sum_{k=1}^{j-1} U_s^k \cdot U_s^{j-k} \\ \beta^j &= -\sum_{k=1}^{j-1} U_s^k \cdot U_t^{j-k} + U_t^k \cdot U_s^{j-k} \\ \gamma^j &= -\sum_{k=1}^{j-1} U_t^k \cdot U_t^{j-k}. \end{cases}$$

Note that the asymptotic direction σ is an eigenvalue of the matrix Q^T with eigenvector $E = \begin{pmatrix} 1 \\ \sigma \end{pmatrix}$. Multiply the system (5.7) by $E^T = (1, \sigma)$ and use the relation $w^j = E^T V^j$ to obtain

$$Lw^j - Aw^j - B\overline{w^j} = E^T K^j$$

with A and B as in the proposition (see [16] for details). It remains to verify that $E^T K^j = -\sum_{k=1}^{j-1} LU^k \cdot LU^{j-k}$. We have

$$\begin{aligned} E^T K^j &= \alpha^j + \sigma\beta^j - \frac{e + 2\sigma f}{g}\gamma^j = \alpha^j + \sigma\beta^j + \sigma^2\gamma^j \\ &= -\sum_{k=1}^{j-1} U_s^k \cdot U_s^{j-k} + \sigma(U_s^k \cdot U_t^{j-k} + U_t^k \cdot U_s^{j-k}) + \sigma^2 U_t^k \cdot U_t^{j-k} \\ &= -\sum_{k=1}^{j-1} LU^k \cdot LU^{j-k} \end{aligned}$$

In the above relations we have used the fact that $g\sigma^2 = -(e + 2f\sigma)$. □

Remark 5.1 Although Proposition 5.1 is stated for the vector field L given in (5.3), a simple calculation shows that equation (5.3) remains valid if L is replaced by any multiple $\widetilde{L} = mL$ and w is replaced by $\widetilde{w} = mw$.

Remark 5.2 In many situations when the degeneracy of the vector field L is not too large, the solvability of (5.6) for the functions $w^1, \dots, w^n$, leads to the determination of the bending fields $U^1, \dots, U^n$. This is the case for surfaces with a flat point (see [17] for the case of homogeneous surfaces). More precisely, if w^1 is known and vanishes at the flat point, then from the relations $LR \cdot U^1 = w^1$, $\overline{L}R \cdot U^1 = \overline{w^1}$, we can solve for two components, say ξ^1 and η^1, of $U^1 = (\xi^1, \eta^1, \zeta^1)$ in terms of w^1, $\overline{w^1}$, and ζ^1. Then we can use $dR \cdot dU^1 = 0$ to reduce it to an integrable system for the component ζ^1 which can be solved by quadrature (Poincaré Lemma). Note also that if w^1 vanishes to order ν at 0, then U^1 can be chosen to vanish at 0 but the order of vanishing is $\nu - 1$. Once U^1 is known, U^2 can also be determined from the knowledge of w^2 and so on.

6. Local non rigidity of a class of surfaces

In this section we consider local deformation of surfaces with positive curvature except at a flat point where both principal curvatures are zero. Assume that $S \subset \mathbb{R}^3$ is a C^∞ surface given as the graph of a function $z(x, y)$. For $\delta > 0$, let

$$S_\delta = \{R(x,y) = (x, y, z(x,y)) \in \mathbb{R}^3 ;\ x^2 + y^2 \le \delta\}. \tag{6.1}$$

We assume that the curvature K of S is positive except at the origin 0 and that z vanishes to order $m > 2$ at 0. Thus,

$$z(x,y) = P_m(x,y) + z_1(x,y) \tag{6.2}$$

where P_m is a polynomial of degree m and where z_1 is a C^∞ function that vanishes to order m at 0. The first theorem establishes the existence of surfaces that are arbitrarily close to S and that are isometric but not congruent (they are not obtained from one another by a rigid motion of $\mathbb{R}^3$).

Theorem 6.1. *Let S be a surface given by* (6.1) *with $K > 0$ except at 0. For every $k \in \mathbb{Z}^+$ and for every $\epsilon > 0$, there exist $\delta > 0$ and surfaces Σ^+ and Σ^- of class C^k over the disc $D(0, \delta)$ such that*

1. *Σ^+ and Σ^- are ϵ-close to S_δ in the C^k-topology;*
2. *Σ^+ and Σ^- are isometric but not congruent.*

This theorem is a consequence of the next theorem about infinitesimal bendings. In the statement of the following theorem, by generic surfaces we mean almost all surfaces. The restriction to generic surfaces is not necessary to reach the conclusion but it is useful for the sake of simplicity in reducing the proof into the setting of solvability of vector fields as in Section 3.

Theorem 6.2. *Let $n \in \mathbb{Z}^+$, $k \in \mathbb{Z}^+$ and let S be a generic surface given by* (6.1) *with curvature $K > 0$ except at 0. Then there exists $\delta > 0$ such that S_δ has nontrivial infinitesimal bendings of order n and class C^k.*

Proof of Theorem 6.1. Given S satisfying the hypotheses of Theorem 6.1, we can find another generic surface $\widetilde{S}$ as in Theorem 6.2 and such that $\widetilde{S}$ is $(\epsilon/2)$-close to

S in the C^k-topology. Let $\widetilde{S}_{t,\delta}$ be an infinitesimal bending of order 1 of class C^k of $\widetilde{S}_\delta$. Thus, $\widetilde{S}_{t,\delta}$ is defined over $D(0,\delta)$ by the position vector

$$\widetilde{R}_t(x,y) = \widetilde{R}(x,y) + tU^1(x,y),$$

where $\widetilde{R}$ is the position vector of $\widetilde{S}_\delta$ and U^1 is a nontrivial bending field. For $t_0 > 0$ define surfaces $\Sigma^\pm$ over the disc $D(0,\delta)$ by the position vectors $\widetilde{R}_{\pm t_0}$. Since $dR \cdot U^1 = 0$, then

$$d\widetilde{R}_{t_0}^2 = d\widetilde{R}_{-t_0}^2 = d\widetilde{R}^2 + t_0^2 dU^1 \cdot dU^1.$$

That is Σ^+ and Σ^- are isometric. They are not congruent because U^1 is nontrivial (see Chapter 12 of [20]). Finally, if t_0 is small enough, $\Sigma^\pm$ are within ϵ from S_δ in the C^k topology. $\square$

Proof of Theorem 6.2.. Let S be defined as the graph of $z(x,y)$ as in (6.2), with $K > 0$ except at 0. In polar coordinates $x = \rho\cos\phi$, $y = \rho\sin\phi$, the function z becomes

$$z = \rho^m P(\phi) + \rho^{m+1} z_2(\rho,\phi)$$

where $P(\phi)$ is a trigonometric polynomial of degree $\leq m$. We can assume that $P > 0$ (S is on one side of the tangent plane at 0). The condition on the curvature means

$$m^2P^2(\phi) + mP(\phi)P''(\phi) - (m-1)P'^2(\phi) > 0 \quad \forall\phi. \tag{6.3}$$

Introduce a new radius ρ_1 by $\rho_1 = \rho P(\phi)^{1/m}$. The surface is given in the coordinates (ρ_1,ϕ) by the position vector

$$R(\rho_1,\phi) = (\rho_1 \mathrm{e}^{q(\phi)}\cos\phi,\ \rho_1 \mathrm{e}^{q(\phi)}\cos\phi,\ \rho_1^m + \rho_1^{m+1} z_3(\rho_1,\phi))$$

where $q(\phi) = \dfrac{-1}{m}\log P(\phi)$. Condition (6.3) can be written for the function q as

$$1 + {q'}^2(\phi) - q''(\phi) > 0,\ \forall\phi.$$

The coefficients of the second fundamental form of S are:

$$\begin{aligned}
e &= \frac{R_{\rho_1\rho_1}\cdot(R_{\rho_1}\times R_\phi)}{|R_{\rho_1}\times R_\phi|} = m(m-1)\rho_1^{m-2} + o(\rho_1^{m-2}),\\
f &= \frac{R_{\rho_1\phi}\cdot(R_{\rho_1}\times R_\phi)}{|R_{\rho_1}\times R_\phi|} = o(\rho_1^{m-1}),\\
g &= \frac{R_{\phi\phi}\cdot(R_{\rho_1}\times R_\phi)}{|R_{\rho_1}\times R_\phi|} = m\rho_1^m(1 + {q'}^2(\phi) - q''(\phi)) + o(\rho_1^m),
\end{aligned}$$

and an asymptotic direction is

$$\sigma = -(f + i\sqrt{eg - f^2}) = -im\rho_1^{m-1}\sqrt{(m-1)(1 + q'^2 - q'')} + o(\rho_1^{m-1}).$$

As vector field of asymptotic directions we can take

$$L = \frac{\partial}{\partial\phi} + i\rho_1 A(\phi,\rho_1)\frac{\partial}{\partial\rho_1},$$

where

$$A(\phi,\rho_1)=\sqrt{\frac{1+q'^2-q''}{m-1}}+o(\rho_1^{m-1}).$$

From now on we will assume that the average $A(\phi,0)$ is not a rational number. That is,

$$c=\frac{1}{2\pi}\int_0^{2\pi}A(\phi,0)d\phi=\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\frac{1+q'^2(\phi)-q''(\phi)}{m-1}}d\phi\ \notin\mathbb{Q}. \tag{6.4}$$

This is the genericity condition that enables us to conjugate the vector field L to a model vector field via C^N-diffeomorphisms for any $N\in\mathbb{Z}^+$ (see section 2). We take as a model vector field

$$L_0=\frac{\partial}{\partial\theta}+icr\frac{\partial}{\partial r}.$$

Furthermore, the diffeomorphism that realizes this conjugacy has the form

$$r=\rho_1+o(\rho_1),\quad \theta=\phi+\beta(\phi)+o(\rho_1)$$

where β is 2π periodic and $\theta'(\phi)>0$. With respect to these new coordinates, the position vector R has the form

$$R=\left(r\mathrm{e}^{q(\mu(\theta))}\cos\mu(\theta)+o(r),\ r\mathrm{e}^{q(\mu(\theta))}\sin\mu(\theta)+o(r),\ r^m+o(r^m)\right),$$

where $\mu(\theta)$ is periodic and $\mu'>0$.

Now, we construct the bending fields $U^1,\dots,U^n$. For this, we construct $w^1,\dots,w^n$, appropriate solutions of the system (5.6), and then deduce the U^j's. With respect to the coordinates (r,θ) and vector field L_0, the system for the w^j's has the form

$$\begin{aligned}L_0w^1&=a(r,\theta)w^1+b(r,\theta)\overline{w^1}\\ L_0w^j&=a(r,\theta)w^j+b(r,\theta)\overline{w^j}-\sum_{k=1}^{j-1}L_0U^k\cdot L_0U^{j-k},\ j\geq 2,\end{aligned} \tag{6.5}$$

where

$$\begin{aligned}a(r,\theta)&=\frac{M''N-MN''}{2(M'N-MN')}+\frac{i}{2}\left(c+\frac{M''N'-M'N''}{c(M'N-MN')}\right)+O(r^{2m-2})\\ b(r,\theta)&=-\frac{M''N-MN''}{2(M'N-MN')}-\frac{i}{2}\left(3c-\frac{M''N'-M'N''}{c(M'N-MN')}\right)+O(r^{2m-2})\end{aligned}$$

with

$$M(\theta)=\mathrm{e}^{q(\mu(\theta))}\cos\mu(\theta)\,,\qquad N(\theta)=\mathrm{e}^{q(\mu(\theta))}\sin\mu(\theta)\,.$$

Note that

$$\mathrm{Re}(a(\theta,0))=\frac{M''N-MN''}{2(M'N-MN')}=-q'(\mu(\theta))\mu'(\theta)+\frac{\mu''(\theta)}{2\mu'(\theta)}$$

and its integral over $[0,\ 2\pi]$ is zero. Thus the above system satisfies the sufficiency conditions for solvability of Section 3. Given $k\in\mathbb{Z}^+$, let w^1 be a nontrivial solution of the first equation of (6.5) such that w^1 vanishes to an order $\nu>k$ along $r=0$

(see Theorem 3.1). To such w^1, corresponds a bending field U^1 of class C^k that vanishes to order $\nu - 1$ on $r = 0$ (see Remark 5.2). With U^1 selected, the equation for w^2 has the nonhomogeneous term $L_0 U^1 \cdot L_0 U^1$ which vanishes to order $(\nu - 1)^2$ on $r = 0$. Again a solution w^2 of the equation in (6.5) can be found that vanishes to an order $\geq \nu$ on $r = 0$ giving rise to a bending field U^2. This process can be continued until we reach U^n. In this way an infinitesimal bending of order n of $\widetilde{S}_\delta$ is achieved. That such a bending is nontrivial follows from the fact that $U^1 \not\equiv 0$ but vanishes to an order $(\nu - 1) > 1$ at 0 and hence cannot be of the form $A \times \widetilde{R}(x, y) + B$ with A and B constants and is not therefore induced by a rigid motion of $\mathbb{R}^3$. □

Remark 6.1 Theorem 6.2 is stated here for surfaces that satisfy condition (6.4). As mentioned earlier this condition is sufficient but is not necessary for the existence of the bending fields. Without appealing to condition (6.4), bendings of homogeneous surfaces are characterized in [17].

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On the Zariski Closure of a Germ of Totally Geodesic Complex Submanifold on a Subvariety of a Complex Hyperbolic Space Form of Finite Volume

Ngaiming Mok

Dedicated to Linda Rothschild on the occasion of her 60th birthday

Abstract. Let X be a complex hyperbolic space form of finite volume, and $W \subset X$ be a complex-analytic subvariety. Let $S \subset X$ be a locally closed complex submanifold lying on W which is totally geodesic with respect to the canonical Kähler-Einstein metric on X. We prove that the Zariski closure of S in W is a totally geodesic subset. The latter implies that the Gauss map on any complex-analytic subvariety $W \subset X$ is generically finite unless W is totally geodesic.

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Keywords. complex hyperbolic space form, holomorphic projective connection, complex geodesic, second fundamental form, holomorphic foliation, Gauss map, Zariski closure, minimal compactification.

A fundamental object of study in projective geometry is the Gauss map. Griffiths-Harris [GH] proved that the Gauss map of a non-linear projective submanifold $W \subset \mathbb{P}^n$ is generically finite. By the work of Zak [Za], the Gauss map on W is in fact a birational morphism onto its image. An elementary proof of the finiteness of the Gauss map on W was given by Ein [Ei]. Regarding the geometry of the Gauss map an analogous study on submanifolds of Abelian varieties was pursued in [Za].

In differential-geometric terms, generic finiteness of the Gauss map on a non-linear projective submanifold $W \subset \mathbb{P}^n$ is the same as the vanishing of the kernel of the projective second fundamental form at a general point. The projective second fundamental form is on the one hand by definition determined by the canonical holomorphic projective connection on the projective space, on the other hand it is

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the same as the restriction to (1,0)-vectors of the Riemannian second fundamental form with respect to the Fubini-Study metric on the projective space. It is therefore natural to extend the study of Gauss maps on subvarieties to the context of subvarieties of complex hyperbolic space forms of finite volume, i.e., subvarieties of quotients of the complex unit ball B^n by torsion-free lattices, either in terms of the canonical holomorphic projective connection on $B^n \subset \mathbb{P}^n$ embedded by means of the Borel embedding, or in terms of the canonical Kähler-Einstein metric, noting that the Riemannian connection of the canonical Kähler-Einstein metric is an affine connection compatible with the canonical holomorphic projective connection on B^n.

In the current article the first motivation is to examine projective geometry in the dual situation of compact complex hyperbolic space forms. Specifically, we will prove generic finiteness of the Gauss map mentioned above on subvarieties $W \subset X$ of compact complex hyperbolic space forms X. Here the Gauss map is defined on regular points of $\pi^{-1}(W)$, where $\pi : B^n \to X$ is the universal covering map, and as such the Gauss map on W is defined only up to the action of Γ, and the question of birationality is not very meaningful. A result on the Gauss map on compact complex hyperbolic space forms analogous to the result of Griffiths-Harris [GH] mentioned in the above is the statement that, given a projective-algebraic submanifold $W \subset X = B^n/\Gamma$ which is not totally geodesic, and taking $\widetilde{W} \subset \pi^{-1}(W)$ to be any irreducible component, the Gauss map on $\widetilde{W}$ is of maximal rank at a general point, or, equivalently, the kernel of the second fundamental form vanishes at a general point. We prove in the current article that this analogue does indeed hold true, and that furthermore, in contrast to the case of projective subvarieties, it holds true more generally for any irreducible complex-analytic subvariety $W \subset X$, *without* assuming that W is nonsingular, with a proof that generalizes to the case of quasi-projective subvarieties of complex hyperbolic space forms of finite volume. We leave unanswered the question whether the analogue of finiteness of the Gauss map for the case of non-linear projective-algebraic submanifolds $W \subset X$ holds true, viz., whether *every* fiber of the Gauss map on $\widetilde{W}$ is necessarily discrete in the case where W is non-singular.

In the case of cocompact lattices $\Gamma \subset \mathrm{Aut}(B^n)$ the result already follows from the study of homomorphic foliations in Cao-Mok [CM, 1990] arising from kernels of the second fundamental form, but we give here a proof that applies to arbitrary holomorphic foliations by complex geodesic submanifolds. Using the latter proof, we are able to study the Zariski closure of a single totally geodesic complex submanifold. For any subset $E \subset W$ we denote by $\mathrm{Zar}_W(E)$ the Zariski closure of E in W. We show that, if the projective variety $W \subset X = B^n/\Gamma$ admits a germ of complex geodesic submanifold $S \subset W$, then some open subset V of $\mathrm{Zar}_W(S)$ in the complex topology, $V \cap S \neq \emptyset$, must admit a holomorphic foliation $\mathcal{F}$ by complex geodesics such $V \cap S$ is saturated with respect to $\mathcal{F}$. Our stronger result on holomorphic foliations by complex geodesic submanifolds implies that $\mathrm{Zar}_W(S)$ must itself be a totally geodesic subset. For the proof of the implication we study varieties of tangents to complex geodesics on a subvariety $W \subset X = B^n/\Gamma$ analogous to the notion of varieties of minimal rational tangents on projective subvarieties uniruled by lines, a notion extensively studied in recent years by Hwang and Mok (cf. Hwang [Hw] and Mok [Mk4]). In a certain sense, total geodesy of the Zariski closure of a germ of complex geodesic submanifold on W results from the algebraicity of varieties of tangents to complex geodesics on W and the asymptotic vanishing of second fundamental forms on locally closed complex submanifolds on B^n swept out by complex geodesics (equivalently minimal disks). This link between the study of subvarieties W of compact complex hyperbolic space forms and those of projective subvarieties is in itself of independent interest, and it points to an approach in the study of Zariski closures of totally geodesic complex submanifolds on projective manifolds uniformized by bounded symmetric domains.

To prove our results for non-uniform lattices $\Gamma \subset \mathrm{Aut}(B^n)$ we make use of the Satake-Borel-Baily compactification ([Sa, 1960], [BB, 1966]) in the case of arithmetic lattices, and the compactification by Siu-Yau [SY, 1982] obtained by differential-geometric means, together with the description of the compactification as given in Mok [Mk3, 2009], in the case of non-arithmetic lattices.

Our result on the Zariski closure of a germ of complex geodesic submanifold in the case of complex hyperbolic space forms is a special case of a circle of problems. In general, we are interested in the characterization of the Zariski closure of a totally geodesic complex submanifold S on a quasi-projective subvariety W of a compact or finite-volume quotient X of a bounded symmetric domain Ω. The inclusion $S \subset X$ is modeled on a pair (D, Ω), where $D \subset \Omega$ is a totally geodesic complex submanifold (which is itself a bounded symmetric domain). The nature and difficulty of the problem may depend on the pair (D, Ω). A characterization of the Zariski closure relates the question of existence of germs of totally geodesic complex submanifolds $S \subset W \subset X$ to that of the global existence of certain types of complex submanifolds $\mathrm{Zar}_W(S) \subset W$. For instance, taking Ω to be biholomorphically the Siegel upper half-plane $\mathcal{H}_g$ of genus $g \geq 2$, $X = \mathcal{H}_g/\Gamma$ to be the moduli space of principally polarized Abelian varieties (where Γ has some torsion), and taking W to be the closure of the Schottky locus, S to be a totally geodesic holomorphic curve, a characterization of $\mathrm{Zar}_W(S)$ probably relates the question of local existence of totally geodesic holomorphic curves to the question of global existence of totally geodesic holomorphic curves and rank-1 holomorphic geodesic subspaces, and hence to a conjecture of Oort's (cf. Hain [Ha]). In such situations, granted the characterization problem on $\mathrm{Zar}_W(S)$ can be settled, a global non-existence result may imply a non-existence result which is local with respect to the complex topology, hence completely transcendental in nature.

1. Statement of the main result and background materials

(1.1) *The Main Theorem on Zariski closures of germs of complex geodesic submanifolds of complex hyperbolic space forms of finite volume.* Let $n \geq 3$ and

$\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice, so that $X := B^n/\Gamma$ is of finite volume with respect to the canonical Kähler-Einstein metric. Let $W \subset X$ be an irreducible complex-analytic subvariety of complex dimension m. A simply connected open subset U of the smooth locus $\mathrm{Reg}(W)$ can be lifted to a locally closed complex submanifold $\widetilde{U}$ on B^n, and on $\widetilde{U}$ we have the Gauss map which associates each point $y \in \widetilde{U}$ to $[T_y(\widetilde{U})]$ as a point in the Grassmann manifold of m-planes in $\mathbb{C}^n$. This way one defines a Gauss map on $\mathrm{Reg}(W)$ which is well defined only modulo the action of the image Φ of $\pi_1(W)$ in $\pi_1(X) = \Gamma$. When the Gauss map fails to be of maximal rank we have an associated holomorphic foliation defined at general points of W whose leaves are totally geodesic complex submanifolds. Partly motivated by the study of the Gauss map on subvarieties of complex hyperbolic space forms we are led to consider the Zariski closure of a single germ of totally geodesic complex submanifold on W. We have

Main Theorem. *Let $n \geq 2$ and denote by $B^n \subset \mathbb{C}^n$ the complex unit ball equipped with the canonical Kähler-Einstein metric $ds^2_{B^n}$. Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice. Denote by $X := B^n/\Gamma$ the quotient manifold, of finite volume with respect to the canonical Kähler-Einstein metric ds^2_X induced from $ds^2_{B^n}$. Denote by $\overline{X}_{\min}$ the minimal compactification of X so that $\overline{X}_{\min}$ is a projective-algebraic variety and X inherits the structure of a quasi-projective variety from $\overline{X}_{\min}$. Let $W \subset X$ be an irreducible quasi-projective subvariety, and $S \subset W$ be a locally closed complex submanifold which is totally geodesic in X with respect to ds^2_X. Then, the Zariski closure $Z \subset W$ of S in W is a totally geodesic subset.*

Here totally geodesic complex submanifolds of X are defined in terms of the canonical Kähler-Einstein metric. Equivalently, they are defined in terms of the canonical holomorphic projective connection on X which descends from $B^n \subset \mathbb{P}^n$ (cf. (1.3)). With the latter interpretation the projective second fundamental form on a locally closed complex submanifold $Z \subset X$ is holomorphic. Total geodesy of Z means precisely the vanishing of the projective second fundamental form.

Consider $X = B^n/\Gamma$, where $\Gamma \subset \mathrm{Aut}(B^n)$ is a non-uniform torsion-free lattice. If Γ is arithmetic, we have the Satake-Borel-Baily compactification (Satake [Sa], Borel-Baily [BB]). For the rank-1 bounded symmetric domain B^n the set of rational boundary components constitute a Γ-invariant subset Π of ∂B^n, and Γ acts on $B^n \cup \Pi$ to give $\overline{X}_{\min} := (B^n \cup \Pi)/\Gamma$, which consists of the union of X and a finite number of points, to be called cusps, such that $\overline{X}_{\min}$ can be endowed naturally the structure of a normal complex space.

(1.2) *Description of Satake-Baily-Borel and Mumford compactifications for $X = B^n/\Gamma$.* We recall briefly the Satake-Baily-Borel and Mumford compactification for $X = B^n/\Gamma$ in the case of a torsion-free non-uniform arithmetic subgroup $\Gamma \subset \mathrm{Aut}\,(B^n)$ (For details cf. Mok [Mk4]). Let $E \subset \partial B^n$ be the set of boundary points b such that for the normalizer $N_b = \{\nu \in \mathrm{Aut}\,(B^n) : \nu(b) = b\}$, $\Gamma \cap N_b$ is an arithmetic subgroup. The points $b \in E$ are the rational boundary components in the sense of Satake [Sa] and Baily-Borel [BB]. Modulo the action of Γ, the set $A = E/\Gamma$ of equivalence classes is finite. Set-theoretically the

Satake-Baily-Borel compactification $\overline{X}_{\min}$ of X is obtained by adjoining a finite number of points, one for each $\alpha \in A$. Fixing $b \in E$ we consider the Siegel domain presentation S_n of B^n obtained via a Cayley transform which maps b to ∞, $S_n = \left\{(z'; z_n) \in \mathbb{C}^n : \mathrm{Im} z_n > \|z'\|^2\right\}$. Identifying B^n with S_n via the Cayley transform, we write $X = S_n/\Gamma$. Writing $z' = (z_1, \ldots, z_{n-1}); z = (z'; z_n)$, the unipotent radical of $W_b \subset N_b$ is given by

$$W_b = \left\{\nu \in N_b : \nu(z'; z_n) = (z' + a'; z_n + 2i\,\overline{a'} \cdot z' + i\|a'\|^2 + t); a' \in \mathbb{C}^{n-1}, t \in \mathbb{R}\right\}, \tag{1}$$

where $\overline{a'} \cdot z' = \sum_{i=1}^{n-1} \overline{a}_i z_i$. W_b is nilpotent and $U_b := [W_b, W_b]$ is the real 1-parameter group of translations $\lambda_t, t \in \mathbb{R}$, where $\lambda_t(z', z) = (z', z + t)$. For a rational boundary component b, $\Gamma \cap W_b \subset W_b$ is a lattice, and $[\Gamma \cap W_b, \Gamma \cap W_b] \subset U_b$ must be nontrivial. Thus, $\Gamma \cap U_b \subset U_b \cong \mathbb{R}$ must be a nontrivial discrete subgroup, generated by some $\lambda_\tau \in \Gamma \cap U_b$. For any nonnegative integer N define

$$S_n^{(N)} = \left\{(z'; z_n) \in \mathbb{C}^n : \mathrm{Im} z_n > \|z'\|^2 + N\right\} \subset S_n\,. \tag{2}$$

Consider the holomorphic map $\Psi : \mathbb{C}^{n-1} \times \mathbb{C} \to \mathbb{C}^{n-1} \times \mathbb{C}^\star$ given by

$$\Psi(z'; z_n) = (z', e^{\frac{2\pi i z_n}{\tau}}) := (w'; w_n)\,; \quad w' = (w_1, \ldots, w_{n-1})\,; \tag{3}$$

which realizes $\mathbb{C}^{n-1} \times \mathbb{C}$ as the universal covering space of $\mathbb{C}^{n-1} \times \mathbb{C}^*$. Write $G = \Psi(S_n)$ and, for any nonnegative integer N, write $G^{(N)} = \Psi(S_n^{(N)})$. We have

$$\widehat{G}^{(N)} = \left\{(w'; w_n) \in \mathbb{C} : |w_n|^2 < e^{\frac{-4\pi N}{\tau}} \cdot e^{\frac{-4\pi}{\tau}\|w'\|^2}\right\}, \quad \widehat{G} = \widehat{G}^{(0)}\;. \tag{4}$$

$\Gamma \cap W_b$ acts as a discrete group of automorphisms on S_n. With respect to this action, any $\gamma \in \Gamma \cap W_b$ commutes with any element of $\Gamma \cap U_b$, which is generated by the translation λ_τ. Thus, $\Gamma \cap U_b \subset \Gamma \cap W_b$ is a normal subgroup, and the action of $\Gamma \cap W_b$ descends from S_n to $S_n/(\Gamma \cap U_b) \cong \Psi(S_n) = G$. Thus, there is a group homomorphism $\pi : \Gamma \cap W_b \to \mathrm{Aut}(G)$ such that $\Psi \circ \nu = \pi(\nu) \circ \Psi$ for any $\nu \in \Gamma \cap W_b$. More precisely, for $\nu \in \Gamma \cap W_b$ of the form (1) where $t = k\tau$, $k \in \mathbb{Z}$, we have

$$\pi(\nu)(w', w_n) = \left(w' + a', e^{-\frac{4\pi}{\tau}\overline{a'} \cdot w' - \frac{2\pi}{\tau}\|a'\|^2} \cdot w_n\right). \tag{5}$$

$S_n/(\Gamma \cap W_b)$ can be identified with $G/\pi(\Gamma \cap W_b)$. Since the action of W_b on S_n preserves ∂S_n, it follows readily from the definition of $\nu(z'; z_n)$ that W_b preserves the domains $S_n^{(N)}$, so that $G^{(N)} \cong S_n^{(N)}/(\Gamma \cap U_b)$ is invariant under $\pi(\Gamma \cap W_b)$. Write $\widehat{G}^{(N)} = G^{(N)} \cup (\mathbb{C}^{n-1} \times \{0\}) \subset \mathbb{C}^n, \widehat{G}^{(0)} = \widehat{G}$. $\widehat{G}^{(N)}$ is the interior of the closure of $G^{(N)}$ in $\mathbb{C}^n$. The action of $\pi(\Gamma \cap W_b)$ extends to $\widehat{G}$. Here $\pi(\Gamma \cap W_b)$ acts as a torsion-free discrete group of automorphisms of $\widehat{G}$. Moreover, the action of $\pi(\Gamma \cap W_b)$ on $\mathbb{C}^{n-1} \times \{0\}$ is given by a lattice of translations Λ_b.

Denote the compact complex torus $(\mathbb{C}^{n-1} \times \{0\})/\Lambda_b$ by T_b. The Mumford compactification $\overline{X}_M$ of X is set-theoretically given by $\overline{X}_M = X \amalg (\amalg T_b)$, the disjoint union of compact complex tori being taken over $A = E/\Gamma$. Define $\Omega_b^{(N)} = \widehat{G}^{(N)}/\pi(\Gamma \cap W_b) \supset G^{(N)}/\pi(\Gamma \cap W_b) \cong S_n^{(N)}/(\Gamma \cap W_b)$. Then the natural map $G^{(N)}/\pi(\Gamma \cap W_b) = \Omega_b^{(N)} - T_b \hookrightarrow S_n/\Gamma = X$ is an open embedding for N sufficiently large, say $N \geq N_0$. The structure of $\overline{X}_M$ as a complex manifold is defined by taking $\Omega_b^{(N)}$, $N \geq N_0$, as a fundamental system of neighborhood of T_b. From the preceding description of $\overline{X}_M$ one can equip the normal bundle $\mathcal{N}_b$ of each compactifying divisor T_b in $\Omega_b^{(N)}(N \geq N_0)$ with a Hermitian metric of strictly negative curvature, thus showing that T_b can be blown down to a normal isolated singularity by Grauert's blowing-down criterion, which gives the Satake-Baily-Borel (*alias* minimal) compactification $\overline{X}_{\min}$. T_b is an Abelian variety since the conormal bundle $\mathcal{N}_b^\star$ is ample on T_b.

In Mok [Mk4] we showed that for a non-arithmetic torsion-free non-uniform lattice $\Gamma \subset \mathrm{Aut}(B^n)$, the complex hyperbolic space form $X = \Omega/\Gamma$ admits a Mumford compactification $\overline{X}_M = X \amalg (\amalg T_b)$, with finitely many Abelian varieties $T_b = \mathbb{C}^n/\Lambda_b$, such that, by collapsing each of the finitely many Abelian varieties T_b, $\overline{X}_M$ blows down to a *projective-algebraic* variety $\overline{X}_{\min}$ with finitely many isolated normal singularities. (As in the statement of the Main Theorem, identifying $X \subset \overline{X}_{\min}$ as a Zariski open subset of the projective-algebraic variety $\overline{X}_{\min}$ we will regard X as a quasi-projective manifold and speak of the latter structure as the canonical quasi-projective structure.) The picture of a fundamental system of neighborhoods $\Omega_b^{(N)}$ of T_b, $N \geq N_0$, in $\overline{X}_M$ is exactly the same as in the arithmetic case. For the topological structure of X, by the results of Siu-Yau [SY] using Busemann functions, to start with we have a decomposition of X into the union of a compact subset $K \subset X$ and finitely many disjoint open sets, called ends, which in the final analysis can be taken to be of the form $\Omega_b^{(N)}$ for some $N \geq N_0$. Geometrically, each end is *a priori* associated to an equivalence class of geodesic rays, where two geodesic rays are said to be equivalent if and only if they are at a finite distance apart from each other. From the explicit description of $\Omega_b^{(N)}$ as given in the above, the space of geodesic rays in each end can be easily determined, and we have

Lemma 1. *Fix $n \geq 1$ and let $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free non-uniform lattice, $X := \Omega/\Gamma$. Let $\overline{X}_M$ be the Mumford compactification and $\Omega_b^{(N)} = \widehat{G}^{(N)}/\pi(\Gamma \cap W_b)$ be a neighborhood of a divisor $T_b = \mathbb{C}^n/\Lambda_b$ at infinity. Denote by $p : G^{(N)} \to G^{(N)}/\pi(\Gamma \cap W_b)$ the canonical projection. Then, any geodesic ray $\lambda : [0, \infty) \to X$ parametrized by arc-length on the end $\Omega_b^{(N)} - T_b = G/\pi(\Gamma \cap W_b) \cong S_n^{(N)}/(\Gamma \cap W_b)$ must be of the form $\lambda(s) = p\left(\Psi(\zeta, a + Aie^{cs})\right)$ for some $\zeta \in \mathbb{C}^{n-1}$, $a \in \mathbb{R}$, some constant $c > 0$ determined by the choice of the canonical Kähler-Einstein metric on B^n and for some sufficiently large constant $A > 0$ (so that $\lambda(0) \in S_n^{(N)}$), where $\Psi(\zeta; \alpha) = (\zeta, e^{\frac{2\pi i \alpha}{\tau}})$, in which $\tau > 0$ and the translation $(z'; z) \to (z', z + \tau)$ is the generator of the infinite subgroup $\Gamma \cap U_b \subset U_b \cong \mathbb{R}$.*

Proof. A geodesic ray on an end $\Omega_b^{(N)} - T_b$ must lift to a geodesic ray on the Siegel domain S_n which converges to the infinity point ∞ of S_n. On the upper half-plane $\mathcal{H} = \{w > 0 : \mathrm{Im}(w) > 0\}$ a geodesic ray $\lambda : [0, \infty) \to \mathcal{H}$ parametrized by arc-length joining a point $w_0 \in \mathcal{H}$ to infinity must be of the form $\mu(s) = u_0 + iv_0 e^{cs}$

for some constant $c > 0$ determined by the Gaussian curvature of the Poincaré metric $ds^2_{\mathcal{H}}$ chosen. In what follows a totally geodesic holomorphic curve on S_n will also be referred to as a complex geodesic (cf. (1.3)). For the Siegel domain $S_n = \{(z'; z_n) \in \mathbb{C}^n : \mathrm{Im} z_n > \|z'\|^2\}$, considered as a fibration over $\mathbb{C}^{n-1}$, the fiber $F_{z'}$ over each $z' \in \mathbb{C}^{n-1}$ is the translate of the upper half-plane by $i\|z\|^2$. $F_{z'} \subset S_n$ is a complex geodesic. It is then clear that any $\nu : [0, \infty) \to S_n$ of the form $\nu(s) = (\zeta, a + Aie^{cs})$, for $\zeta \in \mathbb{C}^{n-1}$ and for appropriate real constants a, c and A, is a geodesic ray on S_n converging to ∞ (which corresponds to $b \in \partial B^n$). Any complex geodesic $S \subset S_n$ is the intersection of an affine line $L := \mathbb{C}\eta + \xi$ with S_n, and S is a disk on L unless the vector η is proportional to $e_n = (0, \ldots, 0; 1)$. Any geodesic ray λ on S_n lies on a uniquely determined complex geodesic S. Thus, $\lambda(s)$ tends to the infinity point ∞ of S_n only if S is parallel in the Euclidean sense to $\mathbb{C}e_n$, so that $\lambda([0, \infty)) \subset F_{z'}$ for some $z' \in \mathbb{C}^{n-1}$. But any geodesic ray on $F_{z'}$ which converges to ∞ must be of the given form $\nu(s) = (\zeta, a + Aie^{cs})$, and the proof of Lemma 1 is complete. □

(1.3) *Holomorphic projective connections.* For the discussion on holomorphic projective connections, we follow Gunning [Gu] and Mok [Mk2]. A holomorphic projective connection Π on an n-dimensional complex manifold X, $n > 1$, consists of a covering $\mathcal{U} = \{U_\alpha\}$ of coordinate open sets, with holomorphic coordinates $(z_1^{(\alpha)}, \ldots, z_n^{(\alpha)})$, together with holomorphic functions $\left({}^\alpha\Phi^k_{ij}\right)_{1 \le i,j,k \le n}$ on U_α symmetric in i, j satisfying the trace condition $\sum_k {}^\alpha\Phi^k_{ik} = 0$ for all i and satisfying furthermore on $U_{\alpha\beta} := U_\alpha \cap U_\beta$ the transformation rule (†)

$$ {}^\beta\Phi^\ell_{pq} = \sum_{i,j,k} {}^\alpha\Phi^k_{ij} \frac{\partial z_i^{(\alpha)}}{\partial z_p^{(\beta)}} \frac{\partial z_j^{(\alpha)}}{\partial z_q^{(\beta)}} \frac{\partial z_\ell^{(\beta)}}{\partial z_k^{(\alpha)}} + \left[\sum_\ell \frac{\partial z_\ell^{(\beta)}}{\partial z_k^{(\alpha)}} \frac{\partial^2 z_k^{(\alpha)}}{\partial z_p^{(\beta)} \partial z_q^{(\beta)}} - \delta^k_p \sigma_q^{(\alpha\beta)} - \delta^k_q \sigma_p^{(\alpha\beta)} \right], $$

where the expression inside square brackets defines the Schwarzian derivative $S(f_{\alpha\beta})$ of the holomorphic transformation given by $z^{(\alpha)} = f_{\alpha\beta}(z^{(\beta)})$, in which

$$ \sigma_p^{(\alpha\beta)} = \frac{1}{n+1} \frac{\partial}{\partial z_p^{(\beta)}} \log J(f_{\alpha\beta}) , $$

$J(f_{\alpha\beta}) = \det\left(\frac{\partial z_i^{(\alpha)}}{\partial z_p^{(\beta)}}\right)$ being the Jacobian determinant of the holomorphic change of variables $f_{\alpha\beta}$. Two holomorphic projective connections Π and Π' on X are said to be equivalent if and only if there exists a common refinement $\mathcal{W} = \{W_\gamma\}$ of the respective open coverings such that for each W_γ the local expressions of Π and Π' agree with each other.

We proceed to relate holomorphic projective connections $\left({}^\alpha\Phi^k_{ij}\right)$ on a complex manifold to affine connections. An affine connection on a holomorphic vector bundle E is said to be a complex affine connection if and only if it is compatible with the Cauchy-Riemann operator $\overline{\partial}_E$ on E, i.e., if and only if covariant differentiation of smooth sections of E against (0,1)-vectors agree with $\overline{\partial}_E$. If a complex manifold X is endowed with a Kähler metric g, then the Riemannian connection of (X, g) extends by change of scalars to the field of complex numbers to a connection on the complexified tangent bundle $T_X^{\mathbb{C}}$, with respect to which $T_X^{\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}$ is a decomposition into a direct sum of parallel smooth complex vector subbundles. When we identify $T_X^{1,0}$ with the holomorphic tangent bundle T_X the restriction of the Riemannian connection to $T_X^{1,0}$ is then a complex affine connection on T_X.

Given a complex manifold X and a complex affine connection ∇ on the holomorphic tangent bundle $T_X = T_X^{1,0}$, by conjugation covariant differentiation is also defined on $\overline{T}_X = T_X^{0,1}$. Thus, ∇ induces a connection on the underlying real manifold X. In what follows by an affine connection on a complex manifold X we will mean a complex affine connection ∇ on the holomorphic tangent bundle T_X. We will say that ∇ is torsion-free to mean that the induced connection on the underlying real manifold X is torsion-free.

Suppose now X is a complex manifold equipped with a holomorphic projective connection. Letting $\big({}^{\alpha}\widetilde{\Gamma}^k_{ij}\big)$ be an affine connection on X, we can define a torsion-free affine connection ∇ on X with Riemann-Christoffel symbols

$$ {}^{\alpha}\Gamma^k_{ij} = {}^{\alpha}\Phi^k_{ij} + \frac{1}{n+1}\sum_{\ell} \delta^k_i \ {}^{\alpha}\widetilde{\Gamma}^{\ell}_{\ell j} + \frac{1}{n+1}\sum_{\ell} \delta^k_j \ {}^{\alpha}\widetilde{\Gamma}^{\ell}_{i\ell}\,. \qquad (\sharp) $$

We say that ∇ is an affine connection associated to Π. Two affine connections ∇ and ∇' on a complex manifold X are said to be projectively equivalent (cf. Molzon-Mortensen [MM, §4]) if and only if there exists a smooth (1,0)-form ω such that $\nabla_\xi\zeta - \nabla'_\xi\zeta = \omega(\xi)\zeta + \omega(\zeta)\xi$ for any smooth (1,0)-vector fields ξ and ζ on an open set of X. With respect to two projectively equivalent affine connections on the complex manifold X, for any complex submanifold S of X, the second fundamental form of the holomorphic tangent bundle T_S as a holomorphic vector subbundle of $T_X|_S$ are the same. In particular, the class of (locally closed) complex geodesic submanifolds S of X are the same. We will say that a (locally closed) complex submanifold $S \subset X$ is geodesic with respect to the holomorphic projective connection Π to mean that it is geodesic with respect to any affine connection ∇ associated to Π. A geodesic 1-dimensional complex submanifold will simply be called a complex geodesic. Associated to a holomorphic projective connection there is a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}T_X$, called the tautological foliation, defined by the lifting of complex geodesics. We have (cf. Mok [Mk, (2.1), Proposition 1])

Lemma 2. *Let X be a complex manifold and $\pi : \mathbb{P}T_X \to X$ be its projectivized holomorphic tangent bundle. Then, there is a canonical one-to-one correspondence between the set of equivalence classes of holomorphic projective connections on X and the set of holomorphic foliations $\mathcal{F}$ on $\mathbb{P}T_X$ by tautological liftings of holomorphic curves.*

From now on we will not distinguish between a holomorphic projective connection and an equivalence class of holomorphic projective connections. Let X be a complex manifold equipped with a holomorphic projective connection Π, ∇ be any affine connection associated to Π by means of ($\sharp$), and $S \subset X$ be a complex submanifold. Then, the second fundamental form σ of T_S in $T_X|_S$ is independent of the choice of ∇. We call σ the projective second fundamental form of $S \subset X$

with respect to Π. Since locally we can always choose the flat background affine connection it follows that the projective second fundamental form is *holomorphic*.

Consider now the situation where X is a complex hyperbolic space form, a complex Euclidean space form, or the complex projective space. X is equipped with a canonical Kähler metric g of constant negative resp. zero resp. positive holomorphic sectional curvature. The universal covering space of X is the complex unit ball B^n resp. the complex Euclidean space $\mathbb{C}^n$ resp. the complex projective space $\mathbb{P}^n$ (itself), equipped with the canonical Kähler-Einstein metric resp. the Euclidean metric resp. the Fubini-Study metric. For $\mathbb{C}^n$ the family of affine lines leads to a tautological foliation $\mathcal{F}_0$ on the projectivized tangent bundle, and g is associated to the flat holomorphic projective connection. In the case of $\mathbb{P}^n$ the projective lines, which are closures of the affine lines in $\mathbb{C}^n \subset \mathbb{P}^n$, are totally geodesic with respect to the Fubini-Study metric g. In the case of $B^n \subset \mathbb{C}^n$, the intersections of affine lines with B^n give precisely the minimal disks which are totally geodesic with respect to the canonical Kähler-Einstein metric g. As a consequence, the tautological foliation $\mathcal{F}$ on $\mathbb{P}T_{\mathbb{P}^n}$ defined by the tautological liftings of projective lines, which is invariant under the projective linear group $\mathrm{Aut}(\mathbb{P}^n) \cong \mathbb{P}\mathrm{GL}(n+1)$, restricts to tautological foliations on $\mathbb{C}^n$ resp. B^n, and they descend to quotients Z of $\mathbb{C}^n$ resp. B^n by torsion-free discrete groups of holomorphic isometries of $\mathbb{C}^n$ resp. B^n, which are in particular projective linear transformations. The holomorphic projective connection on $\mathbb{P}^n$ corresponding to $\mathcal{F}$ will be called the canonical holomorphic projective connection. The same term will apply to holomorphic projective connections induced by the restriction of $\mathcal{F}$ to $\mathbb{C}^n$ and to B^n and to the tautological foliations induced on their quotient manifolds X as in the above. Relating the canonical holomorphic projective connections to the canonical Kähler metric g, we have the following result (cf. Mok [Mk2, (2.3), Lemma 2]).

Lemma 3. *Let (X, g) be a complex hyperbolic space form, a complex Euclidean space form, or the complex projective space equipped with the Fubini-Study metric. Then, the Riemannian connection of the Kähler metric g induces an affine connection on X (i.e., a complex affine connection on T_X) which is associated to the canonical holomorphic projective connection on X. As a consequence, given any complex submanifold $S \subset X$, the restriction to $(1,0)$-vectors of the second fundamental form on $(S, g|_S)$ as a Kähler submanifold of (X, g) agrees with the projective second fundamental form of S in X with respect to the canonical holomorphic projective connection.*

In the notation of Lemma 3, from now on by the second fundamental form σ of S in X we will mean the projective second fundamental form of S in X or equivalently the restriction to (1,0)-vectors of the second fundamental form of $(S, g|_S)$ in (X, g) as a Kähler submanifold. Denoting by h the Hermitian metric on T_X induced by the Kähler metric g, the latter agrees with the second fundamental form of $(T_S, h|_{T_S})$ in $(T_X|_S, h)$ as a Hermitian holomorphic vector subbundle.

We will make use of Lemma 3 to study locally closed submanifolds of complex hyperbolic space forms admitting a holomorphic foliation by complex geodesic submanifolds. The first examples of such submanifolds are given by level sets of the Gauss map. By Lemma 3, it is sufficient to consider the second fundamental form with respect to the flat connection in a Euclidean space, and we have the following standard lemma. (For a proof cf. Mok [Mk1, (2.1), Lemma 2.1.3].)

Lemma 4. *Let $\Omega \subset \mathbb{C}^n$ be a domain and $Z \subset \Omega$ be a closed complex submanifold. At $z \in Z$ denote by $\sigma_z : T_z(Z) \times T_z(Z) \to N_{Z|\Omega,z}$ the second fundamental form with respect to the Euclidean flat connection ∇ on Ω. Denote by $\mathrm{Ker}(\sigma_z) \subset T_z(Z)$ the complex vector subspace of all η such that $\sigma_z(\tau,\eta) = 0$ for any $\tau \in T_z(Z)$. Suppose $\mathrm{Ker}(\sigma_z)$ is of the same positive rank d on Z. Then, the distribution $z \to \mathrm{Re}(\mathrm{Ker}(\sigma_z))$ is integrable and the integral submanifolds are open subsets of d-dimensional affine-linear subspaces.*

With regard to the Gauss map, in the case of projective submanifolds we have the following result of Griffiths-Harris [GH] according to which the Gauss map is generically finite on a non-linear projective submanifold. Expressed in terms of the second fundamental form, we have

Theorem (Griffiths-Harris [GH, (2.29)]). *Let $W \subset \mathbb{P}^N$ be a k-dimensional projective submanifold other than a projective linear subspace. For $w \in W$ denote by $\sigma_w : T_w(W) \times T_w(W) \to N_{W|\mathbb{P}^n,w}$ the second fundamental form in the sense of projective geometry. Then, $\mathrm{Ker}(\sigma_w) = 0$ for a general point $w \in W$.*

In Section 2 we will prove a result which includes the dual analogue of the result above for complex submanifolds of compact complex hyperbolic space forms. As will be seen, smoothness is not essential for the validity of the dual statement.

For the purpose of studying asymptotic behavior of the second fundamental form on certain submanifolds of the complex unit ball B^n we will need the following standard fact about the canonical Kähler-Einstein metric $ds^2_{B^n}$. We will normalize the latter metric so that the minimal disks on B^n are of constant holomorphic sectional curvature -2. With this normalization, writing $z = (z_1, \ldots, z_n)$ for the Euclidean coordinates on $\mathbb{C}^n$, and denoting by $\|\cdot\|$ the Euclidean norm, the Kähler form ω_n of $ds^2_{B^n}$ is given by $\omega_n = i\partial\overline{\partial}(-\log(1-\|z\|^2))$. We have

Lemma 5. *Let $n \geq 1$ and (B^n, ds_{B^n}) be the complex unit n-ball equipped with the canonical Kähler-Einstein metric of constant holomorphic sectional curvature -2. Write $\left(g_{\alpha\overline{\beta}}\right)_{1\leq\alpha,\beta\leq n}$ for the expression of $ds^2_{B^n}$ as Hermitian matrices in terms of the Euclidean coordinates $z = (z_1, \ldots, z_n)$. Let t be a real number such that $0 \leq t \leq 1$. Then, at $(t, 0, \ldots, 0) \in B^n$ we have*

$$g_{1\overline{1}}(t,0,\ldots,0) = \frac{1}{1-t^2}\ ;\ \ g_{\alpha\overline{\alpha}}(t,0,\ldots,0) = \frac{1}{\sqrt{1-t^2}}\ \text{ for }\ 2 \leq \alpha \leq n\,;$$
$$g_{\beta\overline{\gamma}}(t,0,\ldots,0) = 0\ \text{ for }\ \beta \neq \gamma,\ 1 \leq \beta, \gamma \leq n\,.$$

Proof. The automorphism group $\mathrm{Aut}(B^n)$ acts transitively on B^n. Especially, given $0 \le t \le 1$ we have the automorphism $\Psi_t = (\psi_t^1, \ldots, \psi_t^n)$ on B^n defined by

$$\Psi_t(z_1, z_2, \ldots, z_n) = \left(\frac{z+t}{1+tz}, \frac{\sqrt{1-t^2}\, z_2}{1+tz}, \ldots, \frac{\sqrt{1-t^2}\, z_n}{1+tz} \right), \tag{1}$$

which maps 0 to $(t, 0, \ldots, 0)$. We have

$$\frac{\partial \psi_t^1}{\partial z_1}(0) = 1 - t^2 \ ; \ \ \frac{\partial \psi_t^\alpha}{\partial z_\alpha}(0) = \sqrt{1-t^2} \ \text{ for } \ 2 \le \alpha \le n\,;$$
$$\frac{\partial \psi_t^\beta}{\partial z_\gamma}(0) = 0 \ \text{ for } \ \beta \ne \gamma,\, 1 \le \beta, \gamma \le n\,. \tag{2}$$

Since the Kähler form ω_n of $ds^2_{B^n}$ is defined by $\omega_n = i\partial\overline{\partial}(-\log(1 - \|z\|^2)$ we have by direct computation $g_{\alpha\overline{\beta}}(0) = \delta_{\alpha\beta}$, the Kronecker delta. Lemma 5 then follows from the invariance of the canonical Kähler-Einstein metric under automorphisms, as desired. □

Any point $z \in B^n$ is equivalent modulo a unitary transformation to a point $(t, 0, \ldots, 0)$ where $0 \le t \le 1$. If we write $\delta(z) = 1 - \|z\|$ on B^n for the Euclidean distance to the boundary ∂B^n, then Lemma 5 says that, for a point $z \in B^n$ the canonical Kähler-Einstein metric grows in the order $\frac{1}{\delta(z)}$ in the normal direction and in the order $\frac{1}{\sqrt{\delta(z)}}$ in the complex tangential directions.

2. Proof of the results

(2.1) *Total geodesy of quasi-projective complex hyperbolic space forms holomorphic foliated by complex geodesic submanifolds.* Let $X = B^n/\Gamma$ be a complex hyperbolic space form, and denote by $\pi : B^n \to X$ the universal covering map. For a complex-analytic subvariety $W \subset X$ we say that W is non-linear if and only if an irreducible component (thus *any* irreducible component) $\widetilde{W}$ of $\pi^{-1}(W) \subset B^n$ is not the intersection of B^n with a complex affine-linear subspace of $\mathbb{C}^n$. Equivalently, this means that $W \subset X$ is not a totally geodesic subset. As one of our first motivations we were aiming at proving generic finiteness of the Gauss map for non-linear quasi-projective submanifolds of complex hyperbolic space forms $X = B^n/\Gamma$ of finite volume. When the Gauss map on a locally closed complex submanifold of B^n fails to be generically finite, on some neighborhood of a general point of the submanifold we obtain by Lemma 4 a holomorphic foliation whose leaves are complex geodesic submanifolds, equivalently totally geodesic complex submanifolds with respect to the canonical Kähler-Einstein metric. We consider this more general situation and prove first of all the following result which in particular implies generic finiteness of the Gauss map for arbitrary non-linear irreducible quasi-projective subvarieties of complex hyperbolic space forms.

Proposition 1. *Let $n \geq 3$, and $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice, $X := B^n/\Gamma$ be the quotient manifold equipped with the canonical structure as a quasi-projective manifold. Let s, d be positive integers such that $s+d := m < n$. Write $\pi : B^n \to X$ for the universal covering map, and let $W \subset X$ be an irreducible m-dimensional quasi-projective subvariety. Denote by $\widetilde{W}$ an irreducible component of $\pi^{-1}(W)$. Let $x \in \widetilde{W}$ be a smooth point, U be a neighborhood of x on the smooth locus of $\widetilde{W}$, $Z \subset U$ be an s-dimensional complex submanifold, and $D \subset T_U$ be an integrable d-dimensional holomorphic foliation such that T_Z and D are transversal to each other on Z,* i.e., *$T_z(U) = T_z(Z) \oplus D_z$ for every $z \in Z$, and such that the leaves on U of the holomorphic foliation $\mathcal{F}$ defined by D are totally geodesic on B^n. Then, $\widetilde{W} \subset B^n$ is itself totally geodesic in B^n.*

Proof. Assume first of all that $W \subset X$ is compact. We will consider both U and Z as germs of complex submanifolds of W at x, and thus they may be shrunk whenever necessary. Write $z = (z_1, \ldots, z_n)$ for the Euclidean coordinates on $\mathbb{C}^n$. We choose now special holomorphic coordinates $\zeta = (\zeta_1, \ldots, \zeta_m)$ on U at $x \in Z \subset U$, as follows. Let $(\zeta_1, \ldots, \zeta_s)$ be holomorphic coordinates on a neighborhood of x in Z. We may choose $(\zeta_1, \ldots, \zeta_s)$ to be 0 at the point $x \in Z$, and, shrinking Z if necessary, assume that the holomorphic coordinates $(\zeta_1, \ldots, \zeta_s)$ are everywhere defined on Z, giving a holomorphic embedding $f : \Delta^s \xrightarrow{\cong} Z \subset U$. Again shrinking Z if necessary we may assume that there exist holomorphic D-valued vector fields $\eta_1, \ldots, \eta_d$ on Z which are linearly independent everywhere on Z (hence spanning the distribution D along Z). Write $\zeta = (\zeta', \zeta'')$, where $\zeta' := (\zeta_1, \ldots, \zeta_s)$ and $\zeta'' := (\zeta_{s+1}, \ldots, \zeta_m)$. Define now $F : \Delta^s \times \mathbb{C}^d \to \mathbb{C}^n$ by

$$F(\zeta', \zeta'') = f(\zeta') + \zeta_{s+1}\eta_1(\zeta') + \cdots + \zeta_m \eta_d(\zeta') \ . \tag{1}$$

By assumption the leaves of the holomorphic foliation $\mathcal{F}$ defined by D are totally geodesic on the unit ball B^n. Since the totally geodesic complex submanifolds are precisely intersections of B^n with complex affine-linear subspaces of $\mathbb{C}^n$, $F(\zeta)$ lies on the smooth locus of $\widetilde{W}$ for ζ belonging to some neighborhood of $\Delta^s \times \{0\}$. Now F is a holomorphic immersion at 0 and hence everywhere on $\Delta^s \times \mathbb{C}^d$ excepting for ζ belonging to some subvariety $E \subsetneq \Delta^s \times \mathbb{C}^d$. Choose now $\xi = (\xi', \xi'') \in \Delta^s \times \mathbb{C}^d$ such that F is an immersion at ξ and such that $F(\xi) := p \in \partial B^n$, $p = F(\xi)$. Let G be a neighborhood of ξ in $\Delta^s \times \mathbb{C}^d$ and $V = B^n(p; r)$ such that $F|_G : G \to \mathbb{C}^n$ is a holomorphic embedding onto a complex submanifold $S_0 \subset V$. Let σ denote the second fundamental form of the complex submanifold $S := S_0 \cap B^n \subset B^n$. To prove Proposition 1 it is sufficient to show that σ vanishes identically on $S := S_0 \cap B^n$. By Lemma 3 the second fundamental form $\sigma : S^2 T_S \to N_{S|B^n}$ agrees with the second fundamental form defined by the canonical holomorphic projective connection on S, and that in turn agrees with the second fundamental form defined by the flat Euclidean connection on $\mathbb{C}^n$. The latter is however defined not just on S but also on $S_0 \subset V$. Thus, we have actually a holomorphic tensor $\sigma_0 : S^2 T_{S_0} \to N_{S_0|V}$ such that $\sigma = \sigma_0|_S$. Denote by $\|\cdot\|$ the norm on $S^2 T_S^\star \otimes N_{S|V \cap B^n}$ induced by $ds^2_{B^n}$. We claim that, for any point $q \in V \cap \partial B^n$, $\|\sigma(z)\|$ tends to 0 as $z \in S$ tends to q.

By means of the holomorphic embedding $F|_G : G \to S \subset V$ identify σ_0 with a holomorphic section of $S^2T^\star_G \otimes F^*T^\star_V/dF(T_G)$. For $1 \le k \le n$ write $\epsilon_k := F^*\frac{\partial}{\partial z_k}$ and $\nu_k := \epsilon_k \mod dF(T_G)$. Then, writing

$$\sigma_0(\zeta) = \sum_{\alpha,\beta=1}^{n} \sigma^k_{\alpha\beta}(\zeta)\, d\zeta^\alpha \otimes d\zeta^\beta \otimes \nu_k(\zeta) \ , \tag{2}$$

for $\zeta \in G \cap F^{-1}(B^n)$ we have

$$\|\sigma(\zeta)\| \le \sum_{\alpha,\beta=1}^{n} \left|\sigma^k_{\alpha\beta}(\zeta)\right| \|d\zeta^\alpha\| \, \|d\zeta^\beta\| \, \|\nu_k(\zeta)\| \ , \tag{3}$$

where the norms $\|\cdot\|$ are those obtained by pulling back the norms on $T^\star_V$ and on $N_{S|V\cap B^n}$. In what follows we will replace V by $B^n(p;r_0)$ where $0 < r_0 < r$ and shrink G accordingly. Since the holomorphic functions $\sigma^k_{\alpha\beta}(\zeta)$ are defined on a neighborhood of $\overline{G}$, they are bounded on G and hence on $G \cap F^{-1}(B^n)$. From Lemma 5 we have

$$\|dz^k\| \le C_1\sqrt{\delta(F(\zeta))} \tag{4}$$

on $B^n \subset \mathbb{C}^n$. Given $q \in V \cap \partial B^n$, by means of the embedding $F|_G : G \to V \subset \mathbb{C}^n$, the holomorphic coordinates $(\zeta_1, \dots, \zeta_m)$ on V can be completed to holomorphic coordinates $(\zeta_1, \dots, \zeta_m; \zeta_{m+1}, \dots, \zeta_n)$ on a neighborhood of q in B^n. Since F is a holomorphic embedding on a neighborhood of $\overline{G}$, expressing each $d\zeta^\alpha$, $1 \le \alpha \le m$, in terms of $dz^k, 1 \le k \le n$, it follows that

$$\|d\zeta^\alpha\| \le C_2\sqrt{\delta(F(\zeta))} \tag{5}$$

for some positive constant C_2 independent of $\zeta \in G$, where $\delta(z) = 1 - \|z\|$ on B^n. (Although the expression of $d\zeta^\alpha$, $1 \le \alpha \le m$, in terms of dz^k, $1 \le k \le m$, depends on the choice of complementary coordinates $(\zeta_{m+1}, \dots, \zeta_n)$, one can make use of any choice of the latter locally for the estimates.) On the other hand, again by Lemma 5 we have

$$\|\epsilon(\zeta)\| \le C_3\delta(F(\zeta)) \tag{6}$$

for some positive constant C_3. By definition $\|\nu_k(\zeta)\| \le \|\epsilon_k(\zeta)\|$, and the estimates (5) and (6) then yield

$$\|\sigma(\zeta)\| \le C_4 \ , \tag{7}$$

for some positive constant C_4 for $\zeta \in G \cap F^{-1}(B^n)$. For the claim we need however to show that $\|\sigma(\zeta)\|$ converges to 0 as $z = F(\zeta)$ converges to $q \in V \cap \partial B^n$. For this purpose we will use a better estimate for $\nu_k(\zeta)$. Write $q = F(\mu)$, $\mu = (\mu', \mu'')$. Assume that $F(0) = 0$, that for $1 \le i \le d$ we have $\eta_i(\mu') = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the ith position, and that $F(\mu) = (1, 0, \dots, 0)$. We consider now $\zeta_t =$

$(\mu'; t, 0, \ldots, 0)$, $0 < t < 1$, as t approaches to 1. Writing $q_t = F(\zeta_t)$, we have

$$\|\nu_k(\zeta_t)\| = \|\epsilon_k \bmod dF(T_G)\| = \left\| \frac{\partial}{\partial z_k} \bmod T_{q_t}(V) \right\| \\ \leq \left\| \frac{\partial}{\partial z_k} \bmod \mathbb{C}\frac{\partial}{\partial z_1} \right\| \leq C\sqrt{\delta(q_t)}\ . \tag{8}$$

To prove the claim we have to consider the general situation of a point $q \in V \cap \partial B^n$, $q = F(\mu)$, and consider $\zeta \in (\Delta_s \times \mathbb{C}^d) \cap F^{-1}(B^n)$ approaching q. Given such a point $\zeta = (\zeta'; \zeta'')$ there exists an automorphism φ of B^n such that $\varphi(F(\zeta'; 0)) = 0$, $\varphi(F(\zeta)) = (t, 0, \ldots, 0)$ for some $t \in (0, 1)$. Moreover, replacing $\eta_1(\zeta'), \ldots, \eta_d(\zeta')$ by a basis of the d-dimensional complex vector space spanned by these d linearly independent vector fields, we may assume that $\eta_i(\zeta') = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith position. The estimates in (8) then holds true for $\sigma(\zeta)$ at the expense of introducing some constant which can be taken independent of $\zeta \in V \cap B^n$ (noting that G and hence V have been shrunk). As ζ converges to q, t converges to 1 too, and, replacing the estimate (6) for ϵ_k (and hence for ν_k) by the sharper estimate (8) we have shown that $\|\sigma(\zeta)\| \to 0$ as $\zeta \to q$ on $V \cap B^n$, proving the claim. (We have actually the uniform estimate $\|\sigma(\zeta)\| \leq C\sqrt{(\delta(F(z))}$ for some positive constant C on $V \cap B^n$, but this estimate will not be needed in the sequel.)

We proceed now to prove Proposition 1 under the assumption that $\Gamma \subset \mathrm{Aut}(B^n)$ is cocompact. Pick any $q \in V \cap \partial B^n$. Choose a sequence of points $z_k \in V \cap B^n \subset \widetilde{W}$ converging to q, and we have $\|\sigma(z_k)\| \to 0$ as $k \to \infty$. Since $\Gamma \subset \mathrm{Aut}(B^n)$ is cocompact, X is compact, so is $W \subset X$. $W = \widetilde{W}/\Phi$ for some discrete subgroup $\Phi \subset \Gamma$ which stabilizes $\widetilde{W}$ as a set. Thus, there exists a compact subset $K \subset \widetilde{W}$ and elements $\varphi_k \in \Phi$ such that $x_k := \varphi_k^{-1}(z_k) \in K$. Passing to a subsequence if necessary we may assume that $\varphi_k^{-1}(z_k)$ converges to some point $x \in K$. Thus

$$\|\sigma(x)\| = \lim_{k\to\infty} \|\sigma(x_k)\| = \lim_{k\to\infty} \|\sigma(\varphi_k^{-1}(z_k))\| = \lim_{k\to\infty} \|\sigma(z_k)\| = 0\ . \tag{9}$$

Now if $y \in \widetilde{W}$ is any point, for the distance function $d(\cdot, \cdot)$ on $(B^n, ds^2_{B^n})$ we have $d(\varphi_k(y), z_k) = d(\varphi_k(y), \varphi_k(x)) = d(y, x)$. Since $z_k \in V \cap \partial B^n$ converges in $\mathbb{C}^n$ to $q \in V \cap \partial B^n$, from the estimates of the metric $ds^2_{B^n}$ in Lemma 5 it follows readily that $\varphi_k(y)$ also converges to q, showing that $\|\sigma(y)\| = \lim_{k\to\infty} \|\sigma(\varphi_k(y))\| = 0$. As a consequence σ vanishes identically on $\widetilde{W}$, implying that $\widetilde{W} \subset B^n$ is totally geodesic, as desired.

It remains to consider the case $X = B^n/\Gamma$, where $\Gamma \subset \mathrm{Aut}(B^n)$ is a torsion-free non-uniform lattice, and $W \subset X \subset \overline{X}_{\min}$ is a quasi-projective subvariety. Recall that the minimal compactification of X is given by $\overline{X}_{\min} = X \amalg \{Q_1, \ldots, Q_N\}$ where $Q_j, 1 \leq j \leq N$, are cusps at infinity. Renumbering the cusps if necessary, the topological closure $\overline{W}$ of $W \subset \overline{X}_{\min}$ is given by $\overline{W} = W \cup \{Q_1, \ldots, Q_M\}$ for some nonnegative integer $M \leq N$. Pick $q \in V \cap \partial B^n$ and let $(z_k)_{k=1}^{\infty}$ be a sequence of points on $V \cap B^n$ such that z_k converges to q. Recall that $\pi : B^n \to X$ is the universal covering map. Either one of the following alternatives occurs. (a) Passing to a subsequence if necessary $\pi(z_k)$ converges to some point $w \in W$. (b) There exists a cusp Q_ℓ, $1 \le \ell \le M$, such that passing to a subsequence $\pi(z_k)$ converges in $\overline{W}$ to Q_ℓ. In the case of Alternative (a), picking $x \in B^n$ such that $\pi(x) = w$, by exactly the same argument as in the case of cocompact lattices Γ it follows that the second fundamental form σ vanishes on $\widetilde{W}$ and thus $\widetilde{W} \subset B^n$ is totally geodesic. It remains to treat Alternative (b). Without loss of generality we may assume that $\pi(z_k)$ already converges to the cusp Q_ℓ.

We adopt essentially the notation in (1.2) on Mumford compactifications, and modifications on the notation will be noted. Let now $\overline{X}_M$ be the Mumford compactification of X given by $\overline{X}_M = X \amalg (T_1 \amalg \cdots \amalg T_N)$, where each $T_j = \mathbb{C}^{n-1}/\Lambda_j$ is an Abelian variety such that the canonical map $\rho : \overline{X}_M \to \overline{X}_{\min}$ collapses T_j to the cusp Q_j. Here and henceforth we write T_j for T_{b_j}, Λ_j for Λ_{b_j}, etc. For a (closed) subset $A \subset X$ we will denote by $\overline{A}_M$ the topological closure of A in $\overline{X}_M$. Write Ω_j for $\Omega_j^{(N)}$ for some sufficiently large integer N, so that the canonical projection $\mu_j : \Omega_j \to T_j$ realizes Ω_j as a disk bundle over T_j. Write $\Omega_j^0 := \Omega_j - T_j$. Considering each $\pi(z_k), 1 \le k < \infty$, as a point in $X \subset \overline{X}_M$, replacing (z_k) by a subsequence if necessary we may assume that z_k converges to a point $P_\ell \in T_\ell$. Recall that the fundamental group of the bundle Ω_ℓ^0 of puncture disks is a semi-direct product $\Lambda_\ell \times \Psi_\ell$ of the lattice $\Lambda_\ell \cong \mathbb{Z}^{2(n-1)}$ with an infinite cyclic subgroup $\Psi_\ell := \Gamma \cap U_{b_\ell} \subset U_{b_\ell} = [W_{b_\ell}, W_{b_\ell}]$. Let D be a simply connected neighborhood of $P_\ell \in T_\ell$, and define $R := \mu_\ell^{-1}(D) - T_\ell \cong D \times \Delta^\star$ diffeomorphically. Then, $\pi_1(R)$ is infinite cyclic. Without loss of generality we may assume that $\pi(z_k)$ to be contained in the same irreducible component E of $R \cap W$. Consider the canonical homomorphisms $\pi_1(E) \to \pi_1(R) \to \pi_1(\Omega_\ell^0) \to \pi_1(X) = \Gamma$. By the description of the Mumford compactification the homomorphisms $\mathbb{Z} \cong \pi_1(R) \to \pi_1(\Omega_\ell^0)$ and $\pi_1(\Omega_\ell^0) \to \pi_1(X) = \Gamma$ are injective. We claim that the image of $\pi_1(E)$ in $\pi_1(R) \cong \mathbb{Z}$ must be infinite cyclic. For the justification of the claim we argue by contradiction. Supposing otherwise the image must be trivial, and E can be lifted in a univalent way to a subset $\widetilde{E} \subset \widetilde{W} \subset B^n$ by a holomorphic map $h : E \to B^n$. Let E' be the normalization of E, and $\overline{E}'_M$ be the normalization of $\overline{E}_M \subset \overline{X}_M$. Composing h on the right with the normalization $\nu : E' \to E$ we have $h' : E' \to \widetilde{E}$. Since $\widetilde{E} \subset B^n$ is bounded, by Riemann Extension Theorem the map h' extends holomorphically to $h^\sharp : \overline{E}'_M \to \mathbb{C}^n$. Suppose $c \in \overline{E}'_M - E'$ and $h^\sharp(c) = a \in B^n$. Since $\pi(h(e)) = e$ for any $e \in E$ it follows that $\pi(h^\sharp(c)) = c \in \overline{W}_M - W$, contradicting with the definition of $\pi : B^n \to X \subset \overline{X}_M$. We have thus proven that $h^\sharp(\overline{E}'_M) \subset \overline{B^n}$ with $h^\sharp(\overline{E}'_M - E') \subset \partial B^n$, a plain contradiction to the Maximum Principle, proving by contradiction that the image of $\pi_1(E) \to \pi_1(R))$ is infinite cyclic, as claimed. As a consequence of the claim, the image of $\pi_1(E)$ in $\pi_1(X) = \Gamma$ is also infinite cyclic. Factoring through $\pi_1(E) \to \pi_1(W) \to \pi_1(X)$, and recalling that Φ is the image of $\pi_1(W)$ in $\pi_1(X)$, the image of $\pi_1(E)$ in Φ is also infinite cyclic.

Recall that $F : U \to V \subset B^n$ is a holomorphic embedding, $\zeta = (\zeta', \zeta'') \in U$, $q = F(\zeta) \in V \cap \partial B^n$, and assume that Alternative (b) occurs for any sequence of points $z_k \in V \cap B^n$ converging to q. To simplify notations assume without loss of generality that $F(\zeta', 0) = 0$, and define $\lambda_0(t) = F(\zeta', t\zeta'')$ for $t \in [0,1]$. Then, re-parametrizing λ_0 we have a geodesic ray $\lambda(s)$, $0 \le s < \infty$ with respect to $ds^2_{B^n}$ parameterized by arc-length such that $\lambda(s)$ converges to $q \in \partial B^n$ in $\mathbb{C}^n$ as $s \to \infty$. For the proof of Proposition 1, it remains to consider the situation where Alternative (b) occurs for any choice of divergent sequence (z_k), $z_k = \lambda(s_k)$ with $s_k \to \infty$. When this occurs, without loss of generality we may assume that $\pi(\lambda([0,\infty))) \subset \Omega^0_\ell$, which is an end of X. Now by [(1.2), Lemma 1] all geodesic rays in Ω^0_ℓ can be explicitly described, and, in terms of the unbounded realization S_n of B_n they lift as a set to $\{(z^1_0, \ldots, z^{n-1}_0, w_0) : w = u_0 + iv,\, v \ge v_0\}$ for some point $(z^1_0, \ldots, z^{n-1}_0; u_0 + iv_0) \in S_n$. We may choose $E \subset \pi^{-1}(R)$ in the last paragraph to contain the geodesic ray $\pi(\lambda([0,\infty)))$ which converges to the point $P_\ell \in T_\ell$. Now $\Omega^0_\ell = G^{(N)}_\ell / \pi(\Gamma' \cap W_{b_\ell})$ for the domain $G^{(N)}_\ell \subset S^n$ which is obtained as a Cayley transform of B^n mapping some $b_\ell \in \partial B^n$ to ∞. The inverse image $\pi^{-1}(\pi(\lambda([0,\infty)))$ is necessarily a countable disjoint union of geodesic rays R_i, which is the image of a parametrized geodesic ray $\rho_i : [0,\infty) \to B^n$ such that $\lim_{s\to\infty} \rho_i(s) := a_i \in \partial B^n$. For any two of such geodesic rays R_i, R_j there exists $\gamma_{ij} \in \Gamma$ such that $R_i = \gamma_{ij}(R_j)$, hence $\gamma_{ij}(a_j) = a_i$. Now both b_ℓ and q are end points of such geodesic rays on B^n and we conclude that $b_\ell = \gamma(q)$ for some $\gamma \in \Gamma$. In what follows without loss of generality we will assume that q is the same as b_ℓ.

For $q = b_\ell \in \partial B^n$, let $\chi : S_n \to B^n$ be the Cayley transform which maps the boundary ∂S_n (in $\mathbb{C}^n$) to $\partial B^n - \{b_\ell\}$. Write $\chi\big(\widehat{G}^{(N)}_\ell\big) := H_\ell$. Denote by μ a generator of the image of $\pi_1(E)$ in $\pi_1(X) = \Gamma$. Recall that, with respect to the unbounded realization of B^n as the Siegel domain S_n, μ corresponds to an element $\mu' \in U_{b_\ell} \subset W_{b_\ell}$, where W_{b_ℓ} is the normalizer at b_ℓ (corresponding to ∞ in the unbounded realization S_n), and U_{b_ℓ} is a 1-parameter group of translations. Conjugating by the Cayley transform, W_{b_ℓ} corresponds to $W^\flat_{b_\ell}$ whose orbits are horospheres with $b_\ell \in \partial B^n$ as its only boundary point. Thus for any $z \in B^n$, $\mu^i(z)$ converges to b_ℓ as $i \to \infty$. Since F is an immersion at ζ, for $z \in \widetilde{W}$, as has been shown the norm $\|\sigma(z)\|$ of the second fundamental form σ vanishes asymptotically as z approaches q. From the invariance $\|\sigma(z)\| = \|\sigma(\mu^i(z))\|$ and the convergence of $\mu^i(z)$ to q it follows that $\|\sigma(z)\| = 0$. The same holds true for any smooth point z' on $\widetilde{W}$. In fact $\mu^i(z')$ converges to b_ℓ for any point $z' \in B^n$ as the distance $d(\mu^i(z'), \mu^i(z)) = d(z', z)$ with respect to $ds^2_{B^n}$ is fixed (while the latter metric blows up in all directions as one approaches ∂B^n). As a consequence, in any event the second fundamental form σ vanishes identically on $\widetilde{W}$, i.e., $\widetilde{W} \subset B^n$ is totally geodesic, as desired. The proof of Proposition 1 is complete. □

Remarks. For the argument at the beginning of the proof showing that $V \cap B^n$ is asymptotically totally geodesic at a general boundary point $q \in V \cap \partial B^n$ with respect to the canonical Kähler-Einstein metric there is another well-known argument which consists of calculating holomorphic sectional curvature asymptotically, as for instance done in Cheng-Yau [CY]. More precisely, by direct computation it can be shown that for a strictly pseudoconvex domain with smooth boundary, with a strictly plurisubharmonic function defining φ, the Kähler metric with Kähler form $i\partial\overline{\partial}(-\log(-\varphi))$ is asymptotically of constant holomorphic sectional curvature equal to -2, which in our situation is enough to imply the asymptotic vanishing of the norm of the second fundamental form. Here we have chosen to give an argument adapted to the geometry of our special situation where $V \cap \partial B^n$ is holomorphically foliated by complex geodesic submanifolds for two reasons. First of all, it gives an interpretation of the asymptotic behavior of the second fundamental form which is not easily seen from the direct computation. Secondly, the set-up of studying holomorphic foliations by complex geodesic submanifolds in which one exploits the geometry of the Borel embedding $B^n \subset \mathbb{P}^n$ may give a hint to approach the general question of characterizing Zariski closures of totally geodesic complex submanifolds in the case of quotients of bounded symmetric domains.

As an immediate consequence of Proposition 1 and Lemma 4, we have the following result on the Gauss map for finite-volume quotients of the complex unit ball.

Theorem 1. *Let $n \geq 2$, and $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice, $X := B^n/\Gamma$. Equipping $X \subset \overline{X}_{\min}$ with the structure of a quasi-projective manifold inherited from the minimal compactification $\overline{X}_{\min}$, let $W \subset X$ be a quasi-projective subvariety. Denote by $W_0 \subset W$ the smooth locus of W. Let $\widetilde{W_0} \subset B^n$ be an irreducible component (equivalently, a connected component) of $\pi^{-1}(W_0))$. Then, the Gauss map is of maximal rank at a general point of $\widetilde{W_0} \subset B^n$ unless $W \subset X$ is a totally geodesic subset.*

Remarks. When $W \subset X$ is projective, the total geodesy of $\widetilde{W} \subset B^n$ in Theorem 1 already follows from the last part of the proof of Cao-Mok [CM, Theorem 1] (inclusive of Lemma 4.1 and the arguments thereafter). We summarize the argument there, as follows. $\left(B^n, ds^2_{B^n}\right)$ is of constant Ricci curvature $-(n+1)$. An m-dimensional locally closed complex submanifold $S \subset B^n$ is totally geodesic if and only if it is of constant Ricci curvature $-(m+1)$. In general, denoting by Ric_{B^n} resp. Ric_S the Ricci curvature form of $\left(B^n, ds^2_{B^n}\right)$ resp. $\left(S, ds^2_{B^n}\big|_S\right)$, and by $\zeta = (\zeta_1, \ldots, \zeta_m)$ local holomorphic coordinates at $x \in S$, we have $\mathrm{Ric}_S = \frac{m+1}{n+1}\mathrm{Ric}_{B^n} - \rho$, where $\rho(\zeta) = \sum_{\alpha,\beta=1}^n \rho_{\alpha\overline{\beta}}(\zeta) d\zeta^\alpha d\overline{\zeta^\beta}$. In the case of Theorem 1, where $S = \widetilde{W}$ in the notations of Proposition 1, the holomorphic distribution D is given by the kernel of the second fundamental form σ, or equivalently by the kernel of the closed $(1,1)$-form ρ. Choosing the local holomorphic coordinates $\zeta = (\zeta_1, \ldots, \zeta_s; \zeta_{s+1}, \ldots, \zeta_m)$ as in the proof of Proposition 1, from the vanishing of $\rho|_L$ for the restriction of ρ to a local leaf of the holomorphic foliation $\mathcal{F}$ defined by D, it follows that $\rho_{\alpha\overline{\alpha}} = 0$ whenever $\alpha > s$. From $\rho \geq 0$ it follows that $\rho_{\alpha\overline{\beta}} = 0$ for all $\alpha > s$ and for all β $(1 \leq \beta \leq m)$. Coupling with $d\rho = 0$ one easily deduces that $\rho_{\alpha\overline{\beta}}(\zeta) = \rho_{\alpha\overline{\beta}}(\zeta_1, \ldots, \zeta_s)$, so that ρ is completely determined by its restriction $\rho|_Z$ to $Z \subset U$, and the asymptotic vanishing of ρ and hence of σ near boundary points of B^n follows from standard asymptotic estimates of the Kähler-Einstein metric $ds^2_{B^n}$.

(2.2) *Proof of the Main Theorem on Zariski closures of germs of complex geodesic submanifolds.* In the Main Theorem we consider quasi-projective subvarieties W of complex hyperbolic space forms of finite volume. For the proof of the Main Theorem first of all we relate the existence of a germ of complex geodesic submanifold S on W with the existence of a holomorphic foliation by complex geodesics defined on some neighborhood of S in its Zariski closure, as follows.

Proposition 2. *Let $n \geq 3$, and $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice, $X := B^n/\Gamma$, which is endowed with the canonical quasi-projective structure. Let $W \subset X$ be an irreducible quasi-projective variety. Let $S \subset W \subset X$ be a locally closed complex geodesic submanifold of X lying on W. Then, there exists a quasi-projective submanifold $Z \subset W$ such that Z is smooth at a general point of S and such that the following holds true. There is some subset $V \subset Z$ which is open with respect to the complex topology such that V is non-singular, $V \cap S \neq \emptyset$, and there is a holomorphic foliation $\mathcal{H}$ on V by complex geodesics such that for any $y \in V \cap S$, the leaf L_y of $\mathcal{H}$ passing through y lies on S.*

Proof. Replacing W by the Zariski closure of S in W, without loss of generality we may assume that S is Zariski dense in W. In particular, a general point of S is a smooth point of W, otherwise $S \subset \mathrm{Sing}(W) \subsetneq W$, contradicting with the Zariski density of S in W. With the latter assumption we are going to prove Proposition 2 with $Z = W$. Let $x \in W$ and $\alpha \in \mathbb{P}T_x(W)$ be a non-zero tangent vector. Denote by S_α the germ of complex geodesic at x such that $T_x(S_\alpha) = \mathbb{C}\alpha$. Define a subset $A \subset \mathbb{P}T_X|_W$ as follows. By definition a point $[\alpha] \in \mathbb{P}T_x(W)$ belongs to A if and only if the germ S_α lies on W. We claim that the subset $A \subset \mathbb{P}T_X|_W$ is complex-analytic. Let $x_0 \in S \subset W$ be a smooth point and U_0 be a smooth and simply connected coordinate neighborhood of x in W which is relatively compact in W. Recall that $\pi : B^n \to X$ is the universal covering map. Let U be a connected component of $\pi^{-1}(U_0)$ lying on $\widetilde{W}$ and $x \in \widetilde{U}$ be such that $\pi(x) = x_0$. Define $S^\sharp = \pi^{-1}(S) \cap U \subset \widetilde{W}$, which is a complex geodesic submanifold of B^n. Identifying U with U_0, we use the Euclidean coordinates on U as holomorphic coordinates on U_0. Shrinking U_0 and hence U if necessary we may assume that $\pi^{-1}(W) \cap U \subset B^n$ is defined as the common zero set of a finite number of holomorphic functions $f_1, \ldots, f_m$ on U. Then $[\alpha] \in A$ if and only if, in terms of Euclidean coordinates given by $\pi|_U : U \xrightarrow{\cong} U_0$, writing $\alpha = (\alpha_1, \ldots, \alpha_n)$ we have $f_k(x_1 + t\alpha_1, \ldots, x_n + t\alpha_n) = 0$ for all k, $1 \leq k \leq m$, and for any sufficiently small complex number t. Consider only the subset $G \subset T_U$ consisting of non-zero tangent vectors α of length < 1 with respect to $ds^2_{B^n}$. Varying t we have a family of holomorphic functions defined on G whose common zero set descends to a subset $A \cap \mathbb{P}T_U$ in $\mathbb{P}T_U$, showing that $A \cap \mathbb{P}T_U \subset \mathbb{P}T_U$ is a complex-analytic subvariety. Since the base point $x_0 \in W$ is arbitrary, we have shown that $A \subset \mathbb{P}T_X|_W$ is a complex-analytic subvariety.

Assume first of all that X is compact. Recall that $S \subset W \subset X$ is a locally closed complex geodesic submanifold. Obviously $\mathbb{P}T_S \subset A$. Let $A_1 \subset A$ be an irreducible component of A which contains $\mathbb{P}T_S$. Denote by $\lambda : \mathbb{P}T_X \to X$ the canonical projection. Consider the subset $W_1 := \lambda(E_1) \subset W$, which contains S. By the Proper Mapping Theorem, $W_1 \subset W$ is a subvariety. Since $S \subset W_1 \subset W$ and S is Zariski dense in W we must have $W_1 = W$. Thus $W \subset X = B^n/\Gamma$ is an irreducible subvariety in X filled with complex geodesics, a situation which is the dual analogue of the picture of an irreducible projective subvariety $Y \subset B$ uniruled by lines (cf. Hwang [Hw1] and Mok [Mk4]).

More precisely, let $\mathcal{G}$ be the Grassmannian of projective lines in $\mathbb{P}^n$, $\mathcal{G} \cong \mathrm{Gr}(2, \mathbb{C}^{n+1})$, and $\mathcal{K}_0 \subset \mathcal{G}$ be the subset of projective lines $\ell \subset \mathbb{P}^n$ such that $\ell \cap B^n$ is non-empty, and $\mathcal{K} \subset \mathcal{K}_0$ be the irreducible component which contains the set $\mathcal{S}$ of projective lines ℓ whose intersection with $S^\sharp$ contains a non-empty open subset of ℓ. Here $B^n \subset \mathbb{C}^n \subset \mathbb{P}^n$ gives at the same time the Harish-Chandra embedding $B^n \subset \mathbb{C}^n$ and the Borel embedding $B^n \subset \mathbb{P}^n$. Let $\rho : \mathcal{U} \to \mathcal{G}$ be the universal family of projective lines on $\mathbb{P}^n$, and $\rho|_{\mathcal{K}} : \rho^{-1}(\mathcal{K}) \to \mathcal{K}$ be the restriction of the universal family to $\mathcal{K}$. We will also write $\mathcal{U}|_{\mathcal{A}} := \rho^{-1}(\mathcal{A})$ for any subset $\mathcal{A} \subset \mathcal{G}$. By means of the tangent map we identify the evaluation map $\mu : \mathcal{U} \to \mathbb{P}^n$ canonically with the total space of varieties of minimal rational tangents $\mu : \mathbb{P}T_{\mathbb{P}^n} \to \mathbb{P}^n$. Thus, μ associates each $[\alpha] \in \mathbb{P}T_x(\mathbb{P}_n)$ to its base point x. Denote by $\mathcal{D} \subset \mathcal{U}|_{\mathcal{K}}$ the subset defined by $\mathcal{D} := \mathcal{U}|_{\mathcal{K}} \cap \mu^{-1}(B^n)$. Then, there exists some non-empty connected open subset $\mathcal{E} \subset \mathcal{K}$ containing $\mathcal{S}$ such that the image of $\mu(\mathcal{D} \cap \mathcal{U}|_{\mathcal{E}})$ contains a neighborhood $U' \subset U$ of x in $\widetilde{W}$.

The subgroup $\Phi \subset \Gamma$ acts canonically on $\mathcal{U}|_{\mathcal{K}}$ and the quotient $\mathcal{U}|_{\mathcal{K}}/\Phi$ is nothing other than $A_1 \subset \mathbb{P}T_X|_W$. Recall that $A_1 \supset \mathbb{P}T_S$. Let $A_2 \subset A_1$ be the Zariski closure of $\mathbb{P}T_S$ in A_1. Again the image $\lambda(A_2) \subset W$ equals W by the assumption that S is Zariski dense in W. Moreover, a general point $[\alpha] \in \mathbb{P}T_S$, $\alpha \in T_x(S)$, is a smooth point of A_2 and $\lambda|_{A_2} : A_2 \to W$ is a submersion at $[\alpha]$. Fix such a general point $[\alpha_0] \in \mathbb{P}T_x(S)$ and let ℓ_0 be a germ of complex geodesic passing through x such that $T_x(\ell_0) = \mathbb{C}\alpha_0$. Let $H \subset W$ be a locally closed hypersurface passing through x such that $\alpha_0 \notin T_x(H)$. Shrinking the hypersurface H if necessary there exists a holomorphic vector field $\alpha(w)$ on H transversal at every point to H such that $\alpha(x) = \alpha_0$ and $\alpha(x') \in T_{x'}(S)$ for every $x' \in H \cap S$. Then, there is an open neighborhood V of H admitting a holomorphic foliation $\mathcal{H}$ by complex geodesics such that the leaf L_w passing through $w \in H$ obeys $T_w(L_w) = \mathbb{C}\alpha(w)$, and such that, as x' runs over $S \cap H$, the family of leaves $L_{x'}$ sweeps through $V \cap S$. In particular, for $y \in V \cap S$ the leaf L_y lies on $V \cap S$. This proves Proposition 2 in the case where X is compact.

It remains to consider the case where $\Gamma \subset \mathrm{Aut}(B^n)$ is non-uniform, in which case we will make use of the minimal compactification $X \subset \overline{X}_{\min}$ of Satake [Sa] and Baily-Borel [BB]. When $W \subset X = B^n/\Gamma$ is compact the preceding arguments go through without modification. At a cusp $Q_\ell \in \overline{X}_{\min}$ the notion of a complex geodesic submanifold is undefined. In order to carry out the preceding arguments when $W \subset X$ is non-compact so that $\overline{W}$ contains some cusps $Q_\ell \in \overline{X}_{\min}$ we have to work on the non-compact manifolds X and $\mathbb{P}T_X$. In the arithmetic case by [Sa] and [BB] the holomorphic tangent bundle T_X admits an extension to a holomorphic vector bundle E defined on $\overline{X}_{\min}$, and the same holds true for the non-arithmetic case by the description of the ends of X as given in (1.2). For the preceding arguments *a priori* Zariski closures on $X \subset \overline{X}_{\min}$ and on $\mathbb{P}T_X \subset \mathbb{P}E$ have to be taken in the topology with respect to which the closed subsets are complex-analytic subvarieties. We call the latter the analytic Zariski topology.

Nonetheless, since the minimal compactification $\overline{X}_{\min}$ of X is obtained by adding a finite number of cusps, for any irreducible complex-analytic subvariety of $Y \subset X$ of positive dimension by Remmert-Stein Extension Theorem the topological closure $\overline{Y}$ in $\overline{X}_{\min}$ a subvariety. On the other hand, since $\dim(\mathbb{P}E - \mathbb{P}T_X) > 0$, the analogous statement is not *a priori* true for $\mathbb{P}T_X \subset \mathbb{P}E$. However, given any complex-analytic subvariety $Z \subset \mathbb{P}T_X$, by the Proper Mapping Theorem its image $\lambda(Z) \subset X$ under the canonical projection map $\lambda : \mathbb{P}T_X \to X$ is a subvariety. Thus $B := \lambda(Z) \subset X \subset \overline{X}_{\min}$ is quasi-projective. In the preceding arguments in which one takes Zariski closure in the compact case, for the non-compact case it remains the case that the subsets $B \subset X$ on the base manifold are quasi-projective. In the final steps of the arguments in which one obtains a subset $A_2 \subset \mathbb{P}T_X|_W$, $A_2 \supset \mathbb{P}T_S$, such that a general point $[\alpha] \in A_2 \cap \mathbb{P}T_X|_U$ over a neighborhood U of some $x \in S$ in W is non-singular and $\lambda|_{A_2}$ is a submersion at $[\alpha]$, A_2 was used only to produce a holomorphically foliated family of complex geodesics of X over some neighborhood V of $x \in S$ in W, and for that argument it is not necessary for A_2 to be quasi-projective. Thus, the arguments leading to the proof of Proposition 2 in the compact case persist in the general case of $W \subset X = B^n/\Gamma$ for any torsion-free lattice $\Gamma \subset \mathrm{Aut}(B^n)$, and the proof of Proposition 2 is complete. □

Finally, we are ready to deduce the Main Theorem from Proposition 2, as follows.

Proof of the Main Theorem. Recall that $X = \Omega/\Gamma$ is a complex hyperbolic space form of finite volume, $W \subset X \subset \overline{X}_{\min}$ is a quasi-projective subvariety, and $S \subset X$ is a complex geodesic submanifold lying on W. As explained in the proof of Proposition 2, the closure $Z \subset X$ of S in W with respect to the analytic Zariski topology is quasi-projective, so without loss of generality we may replace W by its Zariski closure with respect to the usual Zariski topology and proceed under that convention with proving that W is totally geodesic. By Proposition 2, there exists some $x \in S$ which is a non-singular point on W and some nonsingular open neighborhood V of x in W which admits a holomorphic foliation $\mathcal{F}$ by complex geodesics (such that the leaves of $\mathcal{F}$ passing through any $y \in V$) lies on $S \cap V$. By Proposition 2, $V \subset X$ is totally geodesic. As a consequence, $W \subset X$ is totally geodesic subset, i.e., the non-singular locus W_0 of W is totally geodesic in X, as desired. □

Remarks. In the proof of Proposition 2 we remarked that any irreducible complex-analytic subvariety $Z \subset X$ of positive dimension extends by Remmert-Stein Extension Theorem to a complex-analytic subvariety in $\overline{X}_{\min}$. As a consequence, in the hypothesis of the Main Theorem, in place of assuming $W \subset X$ to be an irreducible quasi-projective subvariety we could have assumed that $W \subset X$ is simply an irreducible complex-analytic subvariety.

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Added in proof

Most recently the author has noticed that a proof of Theorem 1 had already been given in Hwang [Hw2].

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The Large Time Asymptotics of the Entropy

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Abstract. In this note we supply the detailed proof of the entropy asymptotics on manifolds with nonnegative Ricci curvature. We also discuss the possible connections between the large time behavior of the entropy and the existence of harmonic functions.

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Keywords. Heat kernel, entropy, harmonic functions.

In this note, due to several requests after the publication of [N2], as well as new potential applications emerging, we supply the details of the computation on the asymptotics of the entropy stated in [N1], Corollary 4.3, and Proposition 1.1 of [N2]. It is my pleasure to contribute this paper to the birthday of Professor Rothschild.

In this note we assume that (M, g) is a complete Riemannian manifold of dimension n with nonnegative Ricci curvature. We also further assume that it has maximum volume growth. Namely

$$\lim_{r\to\infty} \frac{V_x(r)}{r^n} > 0,$$

where $V_x(r)$ is the volume of the ball of radius r centered at x. Let $H(x, y, t)$ be the heat kernel, or equivalently the minimum positive fundamental solution to the heat operator $\frac{\partial}{\partial t} - \Delta$. For this note we fix x and write simply abbreviate $H(x, y, t)$ as $H(y, t)$, and $V_x(r)$ as $V(r)$. Also we simply denote the distance between x and y by $r(y)$. Recall the definition of the *Perelman's entropy*

$$\mathcal{W}(H, t) := \int_M \left(t|\nabla f|^2 + f - n\right) H\, d\mu$$

where $f = -\log H - \frac{n}{2}\log(4\pi t)$. Also recall the *Nash entropy*

$$\mathcal{N}(H, t) = -\int_M H \log H\, d\mu - \frac{n}{2}\log(4\pi t) - \frac{n}{2}.$$

Let ν_∞ be the cone angle at infinity which can be defined by

$$\nu_\infty := \lim_{r\to\infty} \frac{\theta(r)}{\omega_n}$$

where $\theta(r) := \frac{V(r)}{r^n}$, $V(r)$ is the volume of the ball $B_x(r)$ centered at x and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Recall that in [N2], page 331, there we have proved that

$$\lim_{t\to\infty} \mathcal{W}(H,t) = \lim_{t\to\infty} \mathcal{N}(H,t).$$

Moreover, it was also proved in [N1] that M is of maximum volume growth if and only if $\lim_{t\to\infty} \mathcal{W}(H,t) > -\infty$. The main purpose here is to supply the detail of the following claim.

Theorem 1. *For any $x \in M$,*

$$\lim_{t\to\infty} \mathcal{W}(H,t) = \lim_{t\to\infty} \mathcal{N}(H,t) = \log \nu_\infty.$$

Besides that it is interesting to be able to detect the geometric information such as the volume ratio ν_∞ from the large time behavior of the entropy, there appeared an extra indication of possible uses of such a result. In an very interesting paper [Ka], Kaimanovich defined an entropy $h(\widetilde{M})$ on the space of so-called *minimal Martin boundary* of the positive harmonic functions on the universal/a regular covering space $\widetilde{M}$ of a compact Riemannian manifold. More precisely, let $\mathcal{H}(\widetilde{M})$ be the vector space with the seminorms $\|u\|_K = \sup_K |u|$, where K is a compact subset. Let $\mathcal{K}_p = \{u \,|u \in \mathcal{H}(\widetilde{M}),\, u(p) = 1, u > 0\}$. This is a convex compact subset. Define the minimal Martin boundary of $\widetilde{M}$ by $\partial^*\widetilde{M} = \{u \in \mathcal{K}_p,\ \text{and } u \text{ minimal}\}$. Here $u > 0$ is called minimal if for any nonnegative harmonic function $h \le u$, h must be a multiple of u. This immediately implies the representation formula: for any positive harmonic function f, there exists a Borel measure μ^f on $\partial^*\widetilde{M}$ such that

$$f(x) = \int_{\partial^*\widetilde{M}} u(x)\, d\mu^f(u).$$

In particular, there exists a measure ν corresponding to $f \equiv 1$. Now $u(x)d\nu(u)$ is also a probability measure. The points $x \in \widetilde{M}$ can be identified with the probability measure $u(x)d\nu(u)$ on $\partial^*\widetilde{M}$. The so-called relative entropy is defined to be $\phi(x,y) = -\int_{\partial^*\widetilde{M}} \log\left(\frac{u(x)}{u(y)}\right) u(y)\, d\nu(u)$. By Jensen's inequality it is easy to see that $\phi(x,y) \ge 0$. The Kaimanovich's entropy is defined by averaging $\phi(x,y)$ as following: First check that

$$\frac{1}{\tau}\int_{\widetilde{M}} \phi(x,y)H(\tau,x,y)\, d\mu_{\widetilde{M}}(y)$$

is a function independent of τ and also descends to M. Here $H(\tau,x,y)$ is the heat kernel of $\widetilde{M}$. Then define

$$h(\widetilde{M}) \doteqdot \int_M \left(\frac{1}{\tau}\int_{\widetilde{M}} \phi(x,y)H(\tau,x,y)\, d\mu_{\widetilde{M}}(y)\right) d\mu_M(x).$$

Here we normalize so that $\mu_M(x)$ is a probability measure on M. From the above definition it was proved by Kaimanovich [Ka] (see also [W]) that

$$h(\widetilde{M}) = \int_M \int_{\widetilde{M}} u(x)|\nabla \log u(x)|^2 d\nu(u)\, d\mu(x).$$

Hence the positivity of $h(\widetilde{M})$ implies the existence (in fact ampleness) of nonconstant positive harmonic functions on $\widetilde{M}$. Most interestingly, the Theorem 2 of [Ka] asserts that

$$h(\widetilde{M}) = -\lim_{t\to\infty} \frac{1}{t} \int_{\widetilde{M}} H(t,x,y) \log H(t,x,y)\, d\mu(y)$$

for any x. This is sensational since it is not hard to check for instance that when $\lambda(\widetilde{M})$, the bottom of the L^2-spectrum of the Laplace operator on $\widetilde{M}$ is positive, $h(\widetilde{M}) \geq 4\lambda(\widetilde{M})$ [L]. In fact the above result of Kaimanovich plays a crucial role in [W], where Xiaodong Wang solves a conjecture of Jiaping Wang on characterizing the universal cover of a compact Riemannian manifolds being the hyperbolic space by the information on $\lambda(\widetilde{M})$ and the lower bound of Ricci curvature of M. (See also [Mu] for the related work on Kähler manifolds.) Motivated by the above discussions, mainly the entropy of Kaimanovich [Ka] and its above-mentioned connection with the large time behaviors of the heat kernel and its direct implications on the existence of harmonic functions, we propose the following problem.

Problem 1. *Let M be a complete Riemannian manifold of positive sectional curvature. When does M admit non-constant harmonic functions of polynomial growth?*

Here a harmonic function $u(x)$ is of polynomial growth if there exist positive constants d, C_u such that

$$|u(y)| \leq C_u(1 + r(y))^d.$$

We conjecture that the necessary and sufficient condition is that $\lim_{t\to\infty} \mathcal{N}(H,t) > -\infty$, namely M is of maximum volume growth. Note that for the corresponding problem on holomorphic functions of polynomial growth, the sufficient part has been solved by the author provided that the manifold is Kähler with bounded non-negative bisectional curvature (cf.[N4]).

Now we devote the rest of the paper to the detailed proof of Theorem 1. Under the above notation, let us first recall a result of Li, Tam and Wang [LTW]. A computation similar as below appeared in the earlier paper [N3], pages 935–936.

Theorem 2. [*Li-Tam-Wang*] *Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold with nonnegative Ricci curvature and maximum volume growth. For any $\delta > 0$, the heat kernel of $(\mathcal{M}^n, g)$ satisfies*

$$\frac{\omega_n}{\theta(\delta r(y))}(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{1+9\delta}{4t} r^2(y)\right) \leq H(y,t)$$
$$\leq (1 + C(n,\theta_\infty)(\delta+\beta))\, \frac{\omega_n}{\theta_\infty}(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{1-\delta}{4t} r^2(y)\right),$$

where $\theta_\infty = \lim_{r\to\infty}\theta(r)$,

$$\beta := \delta^{-2n} \max_{r\geq (1-\delta)r(y)} \left(1 - \frac{\theta_x(r)}{\theta_x(\delta^{2n+1}r)}\right).$$

Note that β is a function of $r(y)$ and

$$\lim_{r(y)\to\infty} \beta = 0.$$

Therefore, for any $\epsilon > 0$, there exists a B sufficiently large such that if $r(y) \geq B$ we have

$$\begin{aligned} &\tfrac{\omega_n}{\theta_\infty}(1-\epsilon)(4\pi t)^{-\frac{n}{2}} \exp\left(-\tfrac{1+9\delta}{4t}r^2(y)\right) \leq H(y,t) \\ &\quad \leq (1 + C(n,\theta_\infty)(\delta+\epsilon))\, \tfrac{\omega_n}{\theta_\infty}(4\pi t)^{-\frac{n}{2}} \exp\left(-\tfrac{1-\delta}{4t}r^2(y)\right). \end{aligned} \tag{1}$$

We can also require that $\theta(\delta^{2n+1}r) \leq (1+\epsilon)\theta_\infty$.

The upper estimates: First by the lower estimate of Li-Tam-Wang,

$$\begin{aligned} \mathcal{N}(H,t) &\leq -\int_M H \log\left(\frac{\omega_n}{\theta(\delta r(y))}\right) d\mu + \int_M H\left(\frac{1+9\delta}{4t}r^2(y)\right) d\mu - \frac{n}{2} \\ &= I + II - \frac{n}{2}. \end{aligned}$$

We shall estimate I and II below as in [N3]. Split

$$I = -\int_0^B - \int_B^\infty \left(\int_{\partial B(s)} H \log\left(\frac{\omega_n}{\theta(\delta r(y))}\right) dA\right) ds = I_1 + I_2.$$

It is easy to see that

$$\lim_{t\to\infty} I_1 \leq 0.$$

To compute II_2 we now make use of the lower estimate in (1) to have that

$$\begin{aligned} I_2 &\leq -(1-\epsilon)\frac{\omega_n}{\theta_\infty}(4\pi t)^{-\frac{n}{2}} \int_B^\infty \int_{\partial B(s)} \exp\left(-\frac{1+9\delta}{4t}s^2\right) \log\left(\frac{\omega_n}{\theta(\delta s)}\right) dA\, ds \\ &\leq \log\left(\frac{(1+\epsilon)\theta_\infty}{\omega_n}\right)(1-\epsilon)n\omega_n(4\pi t)^{-\frac{n}{2}} \int_B^\infty \exp\left(-\frac{1+9\delta}{4t}s^2\right) s^{n-1}\, ds. \end{aligned}$$

Here we have used that $\theta(\delta r(y)) \leq \theta_\infty(1+\epsilon)$ and the surface area of $\partial B(s)$ satisfies $A(s) \geq n\theta_\infty s^{n-1}$. Computing the integral via the change of variable $\tau = \frac{1+9\delta}{4t}s^2$ and taking $t\to\infty$ we have that

$$\lim_{t\to\infty} I_2 \leq \log\left(\frac{(1+\epsilon)\theta_\infty}{\omega_n}\right)(1-\epsilon)\,(1+9\delta)^{-n/2}.$$

The estimate of II is very similar. Using the Gamma function identity

$$\Gamma(\frac{n}{2}+1) = \Gamma(\frac{n}{2})\frac{n}{2}$$

we can have that

$$\lim_{t\to\infty} II \leq (1+\epsilon)(1+C(n,\theta_\infty)(\delta+\epsilon))(1+9\delta)(1-\delta)^{n/2+1}\frac{n}{2}.$$

Summarizing we have that

$$\begin{aligned}\lim_{t\to\infty}\mathcal{N}(H,t) &\le \log\left(\frac{(1+\epsilon)\theta_\infty}{\omega_n}\right)(1-\epsilon)\,(1+9\delta)^{-n/2}\\ &\quad +(1+\epsilon)(1+C(n,\theta_\infty)(\delta+\epsilon))(1+9\delta)(1-\delta)^{n/2+1}\frac{n}{2}-\frac{n}{2}.\end{aligned}$$

Letting $\epsilon\to 0$, then $\delta\to 0$ we have what

$$\lim_{t\to\infty}\mathcal{N}(H,t)\le\log\nu_\infty.$$

For the lower estimate, we use the other inequality provided by Theorem 2, mainly (1). First write

$$\begin{aligned}\mathcal{N}(H,t) &\ge \int_M H\left(\frac{1-\delta}{4t}\right)r^2(y)\,d\mu-\log\left[(1+C(n,\theta_\infty)(\delta+\epsilon))\,\frac{\omega_n}{\theta_\infty}\right]-\frac{n}{2}\\ &= \int_{r\le B}+\int_{r\ge B} H\left(\frac{1-\delta}{4t}\right)r^2(y)\,d\mu-\log\left[(1+C(n,\theta_\infty)(\delta+\epsilon))\,\frac{\omega_n}{\theta_\infty}\right]-\frac{n}{2}.\end{aligned}$$

Similarly

$$\lim_{t\to\infty}\int_{r\le B}H\left(\frac{1-\delta}{4t}\right)r^2(y)\,d\mu\to 0$$

as $t\to\infty$. On the other hand, using that $A(r)\ge n\theta_\infty r^{n-1}$,

$$\begin{aligned}\lim_{t\to\infty}I_3 &\doteq \int_{r\ge B}H\left(\frac{1-\delta}{4t}\right)r^2(y)\,d\mu\\ &\ge \lim_{t\to\infty}n\omega_n(1-\epsilon)(1-\delta)\frac{1}{(4\pi t)^{n/2}}\int_B^\infty\exp\left(-\frac{(1+9\delta)r^2}{4t}\right)\frac{r^2}{4t}r^{n-1}\,dr.\end{aligned}$$

The direct calculation shows that

$$\lim_{t\to\infty}\frac{1}{(4\pi t)^{n/2}}\int_B^\infty\exp\left(-\frac{(1+9\delta)r^2}{4t}\right)\frac{r^2}{4t}r^{n-1}\,dr=\frac{1}{2}\Gamma(\frac{n}{2}+1)(1+9\delta)^{-\frac{n}{2}-1}.$$

Using $\omega_n=\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})n}=\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, we finally have that

$$\lim_{t\to\infty}I_3\ge\frac{n}{2}(1-\epsilon)(1-\delta)(1+9\delta)^{-\frac{n}{2}-1}.$$

The lower estimate

$$\lim_{t\to\infty}\mathcal{N}(H,t)\ge\log\nu_\infty$$

follows after taking $\epsilon\to 0$ and then $\delta\to 0$.

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The Closed Range Property for $\overline{\partial}$ on Domains with Pseudoconcave Boundary

Mei-Chi Shaw

Dedicated to Professor Linda P. Rothschild

Abstract. In this paper we study the $\bar{\partial}$-equation on domains with pseudoconcave boundary. When the domain is the annulus between two pseudoconvex domains in $\mathbb{C}^n$, we prove L^2 existence theorems for $\bar{\partial}$ for any $\bar{\partial}$-closed (p,q)-form with $1 \le q < n-1$. We also study the critical case when $q = n-1$ on the annulus Ω and show that the space of harmonic forms is infinite dimensional. Some recent results and open problems on pseudoconcave domains in complex projective spaces are also surveyed.

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1. Introduction

In the Hilbert space approach, the closed range property for an unbounded closed operator characterizes the range of the operator. Thus it is important to know whether the range of an unbounded operator is closed. When the unbounded operator is the Cauchy-Riemann equation, the Hilbert space approach has been established by the pioneering work of Kohn [Ko1] for strongly pseudoconvex domains and by Hörmander [Hör1] for pseudoconvex domain in $\mathbb{C}^n$ or a Stein manifold. The following L^2 existence and regularity theorems for $\bar{\partial}$ on pseudoconvex domains in $\mathbb{C}^n$ (or a Stein manifold) are well known.

Theorem (Hörmander [Hör1]**).** *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain. For any $f \in L^2_{(p,q)}(\Omega)$, where $0 \le p \le n$ and $1 \le q < n$, such that $\bar{\partial} f = 0$ in Ω, there exists*

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$u \in L^2_{(p,q-1)}(\Omega)$ satisfying $\bar\partial u = f$ and

$$\int_\Omega |u|^2 \le \frac{e\delta^2}{q}\int_\Omega |f|^2$$

where δ is the diameter of Ω.

This implies that the range of $\bar\partial$ is equal to the kernel of $\bar\partial$, which is closed since $\bar\partial$ is a closed operator. It also follows that the harmonic forms are trivial for $1 \le q \le n$. Furthermore, if the boundary $b\Omega$ is smooth, we also have the following global boundary regularity results for $\bar\partial$.

Theorem (Kohn [Ko2]**).** *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary $b\Omega$. For any $f \in W^s_{(p,q)}(\Omega)$, where $s \ge 0$, $0 \le p \le n$ and $1 \le q < n$, such that $\bar\partial f = 0$ in Ω, there exists $u \in W^s_{(p,q)}(\Omega)$ satisfying $\bar\partial u = f$.*

In this paper we study the $\bar\partial$-equation on domains with pseudoconcave boundary. When the domain is the annulus between two pseudoconvex domains in $\mathbb{C}^n$, the closed range property and boundary regularity for $\bar\partial$ were established in the author's earlier work [Sh1] for $0 < q < n-1$ and $n \ge 3$. In this paper, we will study the critical case when $q = n-1$ on the annulus Ω. In this case the space of harmonic forms is infinite dimensional. We also show that in the case when $0 < q < n-1$, the space of harmonic forms is trivial. This improves the earlier results in [Sh1] where only finite dimensionality for the harmonic forms has been established. We first study the closed range property for the case when the annulus is between two strictly pseudoconvex domains in Section 2. This simpler case warrants special attention since it already illuminates the difference between $q < n-1$ and $q = n-1$. Then we study the case when the annulus is between two weakly pseudoconvex domains in Section 3. Special attention is given to the case when $n = 2$ and $q = 1$. In Section 4 we survey some known existence and regularity results for $\bar\partial$ on pseudoconcave domains with Lipschitz domains in the complex projective space $\mathbb{C}P^n$ when $n \ge 3$. The closed range property for $\bar\partial$ for $(0,1)$-forms with L^2 coefficients on pseudoconcave domains in $\mathbb{C}P^n$ is still an open problem (see Conjecture 1 and Conjecture 2 at the end of the paper). Very little is known on the pseudoconcave domains in $\mathbb{C}P^2$.

The author would like to thank professor Lars Hörmander who first raised this question to the author on the closed range property for $\bar\partial$ on the annulus for the critical degree. This paper is greatly inspired by his recent paper [Hör2]. She would also like to thank professor Emil Straube for his comments on the proof of Theorem 3.2.

2. The $\bar\partial$-equation on the annulus between two strictly pseudoconvex domains in $\mathbb{C}^n$

Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ between two strictly pseudoconvex domains $\Omega_2 \subset\subset \Omega_1$ with smooth boundary. In this section, we study the L^2

existence for $\bar\partial$ on Ω. We first prove the L^2 existence theorem of the $\bar\partial$-Neumann operator for the easier case when $q < n-1$. Let $\bar\partial^*$ be the Hilbert space adjoint of $\bar\partial$. As before, we formulate the $\bar\partial$-Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and the harmonic space of (p,q)-forms $\mathcal{H}_{(p,q)}$ is defined as the kernel of $\Box$.

Theorem 2.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ between two strictly pseudoconvex domains $\Omega_2 \subset\subset \Omega_1$ with smooth boundary.*

The $\bar\partial$-Neumann operator $N(p,q)$ exists and $N(p,q) : L^2_{(p,q)}(\Omega) \to W^1_{(p,q)}(\Omega)$, where $0 \le p \le n$, $1 \le q \le n-2$. The space of harmonic space $\mathcal{H}_{(p,q)}$ is finite dimensional.

Proof. Recall that condition $Z(q)$ means that the Levi form has at least $n-q$ positive eigenvalues or $q+1$ negative eigenvalues. From our assumption, the Levi form for the boundary $b\Omega_2$ has $n-1$ negative eigenvalues at each boundary point. Thus it satisfies condition $Z(q)$ for $0 \le q < n-1$. The Levi form on $b\Omega_1$ satisfies condition $Z(q)$ for $0 < q < n$. Thus we have the estimates (see [Hör1], [FK] or [CS])

$$\|f\|^2_{\frac{1}{2}} \le C(\|\bar\partial f\|^2 + \|\bar\partial^* f\|^2 + \|f\|^2), \quad f \in L^2_{(p,q)}(\Omega) \cap \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*).$$

This gives the existence of the $\bar\partial$-Neumann operator $N_{(p,q)} : L^2_{(p,q)}(\Omega) \to W^1_{(p,q)}(\Omega)$ and the harmonic space $\mathcal{H}_{(p,q)}$ is finite dimensional. $\Box$

Theorem 2.2. *Let Ω be the same as in Theorem 2.1. For any $\bar\partial$-closed $f \in L^2_{(p,q)}(\Omega) \cap \mathrm{Ker}(\bar\partial)$, where $0 \le p \le n$, $1 \le q \le n-2$, there exists $u \in W^{\frac{1}{2}}_{(p,q-1)}(\Omega)$ with $\bar\partial u = f$. The space of harmonic (p,q)-forms $\mathcal{H}_{(p,q)}$ is trivial when $1 \le q \le n-2$.*

Proof. From Theorem 2.1, for any $f \in L^2_{(p,q)}(\Omega) \cap \mathrm{Ker}(\bar\partial)$ and $f \perp \mathcal{H}_{(p,q)}$, there exists a $u \in W^{\frac{1}{2}}_{(p,q-1)}(\Omega)$ such that $\bar\partial u = f$.

To show that the harmonic forms are trivial, notice that from the regularity of the $\bar\partial$-Neumann operator in Theorem 2.1, the harmonic forms are smooth up to the boundary. Also to prove Theorem 2.2, it suffices to prove the *a priori* estimates for smooth $\bar\partial$-closed forms f since they are dense.

Let $N^2_{(p,q)}$ denote the $\bar\partial$-Neumann operator on the strongly pseudoconvex domain Ω_2. For any smooth $\bar\partial$-closed $f \in C^\infty_{(p,q)}(\overline{\Omega})$, we first extend f from Ω smoothly to $\tilde f$ in Ω_1. We have that

$$\|\bar\partial\tilde f\|_{W^{-1}_{(p,q+1)}(\Omega_2)} \le C\|\tilde f\|_{L^2_{(p,q)}(\Omega_1)} \le C\|f\|_{L^2_{(p,q)}(\Omega)}.$$

Let $v = -\star\bar\partial N^{\Omega_2}_{(n-p,n-q-1)} \star \bar\partial\tilde f$. We have $\bar\partial v = \bar\partial\tilde f$ in the distribution sense in Ω_1 if we extend v to be zero outside Ω_2 since $q < n-1$ (see Theorem 9.1.2 in the book by Chen-Shaw [CS]). Also v satisfies

$$\|v\|_{W^{-\frac{1}{2}}(\Omega_2)} \le C\|\bar\partial\tilde f\|_{W^{-1}_{(p,q+1)}(\Omega_2)}.$$

Setting $F = \tilde{f} - v$, we have that $\bar{\partial}F = 0$ in Ω_1 and $F = f$ on Ω. Thus we have extended f as a $\bar{\partial}$-closed form F in Ω_1 and

$$\|F\|_{W^{-\frac{1}{2}}_{(p,q)}(\Omega_1)} \le C\|f\|_{L^2_{(p,q)}(\Omega)}.$$

Thus $F = \bar{\partial}U$ with $U \in W^{\frac{1}{2}}(\Omega_1)$. Setting $u = U\,|_{\Omega}$, we have $\bar{\partial}u = f$ with

$$\|u\|_{W^{\frac{1}{2}}_{(p,q-1)}(\Omega)} \le C\|f\|_{L^2_{(p,q)}(\Omega)}.$$

For general $\bar{\partial}$-closed $f \in L^2_{(p,q)}(\Omega)$, we approximate f by smooth forms to obtain a solution $u \in W^{\frac{1}{2}}_{(p,q-1)}(\Omega)$ satisfying $\bar{\partial}u = f$.

If $f \in \mathcal{H}_{(p,q)}(\Omega)$, we have $\bar{\partial}f = \bar{\partial}^*f = 0$ in Ω. This means that $f = \bar{\partial}u$ and

$$\|f\|^2 = (\bar{\partial}u, \bar{\partial}u) = (u, \bar{\partial}^*f) = 0.$$

Thus we have $\mathcal{H}_{(p,q)}(\Omega) = \{0\}$. □

Next we discuss the case when $q = n-1$, where condition $Z(q)$ is not satisfied on $b\Omega_2$.

Theorem 2.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ between two strictly pseudoconvex domains $\Omega_2 \subset\subset \Omega_1$ with smooth boundary, where $n \ge 3$. For any $0 \le p \le n$, the range of $\bar{\partial} : L^2_{(p,n-2)}(\Omega) \to L^2_{(p,n-1)}(\Omega)$ is closed and the $\bar{\partial}$-Neumann operator $N_{(p,n-1)}$ exists on $L^2_{(p,n-1)}(\Omega)$. For any $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) \cap (\mathcal{H}_{(p,n-1)})^\perp$, we have*

$$\|f\|^2_{\frac{1}{2}} \le C(\|\bar{\partial}f_1\|^2 + \|\bar{\partial}^*f_2\|^2). \tag{2.1}$$

Moreover, for any $f \in L^2_{(p,n-1)}(\Omega)$,

$$\|N_{(p,n-1)}f\|_{\frac{1}{2}} \le C\|f\|,$$

$$\|\bar{\partial}N_{(p,n-1)}f\|_{\frac{1}{2}} + \|\bar{\partial}^*N_{(p,n-1)}f\|_{\frac{1}{2}} \le C\|f\|.$$

Proof. We note that for any domain, one can always solve $\bar{\partial}$ for the top degree and condition $Z(n)$ is a void condition. When $n \ge 3$, Ω satisfies both condition $Z(n-2)$ and $Z(n)$, thus from Theorem (3.1.19) in [FK], the $\bar{\partial}$-equation has closed range and $\bar{\partial}$-Neumann operator exists for $(p, n-1)$-forms. For $f \in L^2_{(p,n-1)}(\Omega)$, we have

$$f = \bar{\partial}\bar{\partial}^*N_{(p,n-1)}f + \bar{\partial}^*\bar{\partial}N_{(p,n-1)}f + H_{(p,n-1)}f$$

where $H_{(p,n-1)}f$ is the projection onto the harmonic space $\mathcal{H}_{(p,n-1)}$.

We note that for any $(p, n-1)$-form $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) \cap (\mathcal{H}_{(p,n-1)})^\perp$, we write $f = f_1 + f_2$ with $f_1 \perp \mathrm{Ker}(\bar{\partial})$ and $f_2 \perp \mathrm{Ker}(\bar{\partial}^*)$. Then $\bar{\partial}f = \bar{\partial}f_1$ and $\bar{\partial}^*f = \bar{\partial}^*f_2$ and we have (see Proposition 3.1.18 in [FK])

$$f = \bar{\partial}^*N_{(p,n)}\bar{\partial}f_1 + \bar{\partial}N_{(p,n-2)}\bar{\partial}^*f_2. \tag{2.2}$$

Using the regularity for $N_{(p,n)}$ and $N_{(p,n-2)}$, we have

$$\begin{aligned}\|f\|^2_{\frac{1}{2}} &\le 2(\|\bar\partial^* N_{(p,n)}\bar\partial f_1\|^2_{\frac{1}{2}} + \|\bar\partial N_{(p,n-2)}\bar\partial^* f_2\|^2_{\frac{1}{2}}) \\ &\le C(\|\bar\partial f_1\|^2 + \|\bar\partial^* f_2\|^2) \\ &= C((\bar\partial f, \bar\partial f) + (\bar\partial^* f, \bar\partial^* f)). \end{aligned} \tag{2.3}$$

If we assume that, in addition, f is in Dom($\Box$), then

$$\|f\|^2_{\frac{1}{2}} \le C((\bar\partial f, \bar\partial f) + (\bar\partial^* f, \bar\partial^* f)) = C(\Box f, f) \le C\|\Box f\|\|f\|. \tag{2.4}$$

Setting $f = N_{(p,n-1)}\phi$ for some $\phi \in L^2_{(p,n-1)}(\Omega)$, we have from (2.3) that

$$\|N\phi\|^2_{\frac{1}{2}} \le C\|\Box N\phi\|\|N\phi\| \le C^2\|\phi\|^2. \tag{2.5}$$

Thus the operator $N_{(p,n-1)}$ is a bounded operator from L^2 to $W^{\frac{1}{2}}$. To show that $\bar\partial N_{(p,n-1)}$ and $\bar\partial^* N_{(p,n-1)}$ is bounded from L^2 to $W^{\frac{1}{2}}$, we use the fact that (2.1) holds for both $q = n-2$ and $q = n$. Substituting f in (2.1) by $\bar\partial N_{(p,n-1)}f$ and $\bar\partial^* N_{(p,n-1)}f$, then we have

$$\|\bar\partial^* Nf\|^2_{\frac{1}{2}} \le C\|\bar\partial\bar\partial^* Nf\|^2, \quad f \in L^2_{(p,n-1)}(\Omega), \tag{2.6}$$

$$\|\bar\partial Nf\|^2_{\frac{1}{2}} \le C\|\bar\partial^*\bar\partial Nf\|^2, \quad f \in L^2_{(p,n-1)}(\Omega). \tag{2.7}$$

Adding (2.6) and (2.7), we get for any $f \in L^2_{(p,n-1)}(\Omega)$,

$$\begin{aligned}\|\bar\partial^* Nf\|^2_{\frac{1}{2}} + \|\bar\partial Nf\|^2_{\frac{1}{2}} &\le C(\|\bar\partial\bar\partial^* Nf\|^2 + \|\bar\partial^*\bar\partial Nf\|^2) \\ &\le C\|\Box Nf\|^2 \le C\|f\|^2. \end{aligned} \tag{2.8}$$

This proves the theorem. $\Box$

Corollary 2.4. *For any $0 \le p \le n$ and $n \ge 3$, the range of $\bar\partial : L^2_{(p,n-2)}(\Omega) \to L^2_{(p,n-1)}(\Omega)$ is closed. For any $f \in L^2_{(p,n-1)}(\Omega)$ with $\bar\partial f = 0$ and $f \perp \mathcal{H}_{(p,n-1)}(\Omega)$, there exists $u \in W^{\frac{1}{2}}_{(p,n-2)}(\Omega)$ satisfying $\bar\partial u = f$.*

Corollary 2.5. *The space of harmonic $(p, n-1)$-forms $\mathcal{H}_{(p,n-1)}$ is infinite dimensional.*

Proof. From Corollary 2.3 and Theorem 2.2, we have that the range of $\Box_{(p,n-1)}(\Omega)$ is closed. The infinite dimensionality of the null space of $\Box_{(p,n-1)}(\Omega)$ is proved in Theorem 3.1 in Hörmander [Hör2]. We will give another proof in Corollary 3.4 in the next section. $\Box$

Remarks. (1) The case for the closed range property on the annulus domains when $n = 2$ and $q = 1$ is more involved (see Theorem 3.3 in the next section).

(2) When the domain Ω is the annulus between two concentric balls or ellipsoids, the infinite-dimensional space $\mathcal{H}_{(p,n-1)}$ has been computed explicitly by Hörmander (see Theorem 2.2 in [Hör2]). For integral formula when $1 \le q < n-1$ in this case, see Section 3.5 in the book by Range [Ra] (see also the paper by Hortmann [Hor]).

3. The $\bar\partial$-equation on the annulus between two weakly pseudoconvex domains in $\mathbb{C}^n$

We recall the following existence and estimates for $\bar\partial$ in the annulus between two pseudoconvex domains (see Shaw [Sh1])

Theorem 3.1. *Let $\Omega \subset\subset \mathbb{C}^n$, $n \geq 3$, be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega_2}$ between two pseudoconvex domains Ω_1 and Ω_2 with smooth boundary and $\Omega_2 \subset\subset \Omega_1$. Let Let ϕ_t be a smooth function which is equal to $t|z|^2$ near $b\Omega_1$ and $-t|z|^2$ near $b\Omega_2$ where $t > 0$. Then for $0 \leq p \leq n$ and $1 \leq q < n-1$, the $\bar\partial$-Neumann operator $N^t_{(p,q)}$ with weights ϕ_t exists for sufficiently large $t > 0$ on $L^2_{(p,q)}(\Omega)$ and the harmonic space $\mathcal{H}_{(p,q)}(\Omega)$ is finite dimensional. For any $f \in L^2_{(p,q)}(\Omega)$, we have*

$$f = \bar\partial\bar\partial^*_t N^t_{(p,q)} f + \bar\partial^*_t \bar\partial N^t_{(p,q)} f + H^t_{(p,q)} f \tag{3.1}$$

where $H^t_{(p,q)} f$ is the projection of f into $\mathcal{H}_{(p,q)}(\Omega)$.

Furthermore, for each $s \geq 0$, there exists T_s such that $N_{(p,q)}$, $\bar\partial^*_t N^t_{(p,q)}$, $\bar\partial N^t_{(p,q)}$ and the weighted Bergman projection $B^t_{(p,0)} = I - \bar\partial^*_t N^t_{(p,1)} \bar\partial$ are exact regular on $W^s_{(p,q)}(\Omega)$ for $t > T_s$.

The existence and regularity of $N^t_{(p,q)}$ was proved in [Sh1]. The exact regularity for the related operators $\bar\partial^*_t N^t_{(p,q)}$, $\bar\partial N^t_{(p,q)}$ and the weighted Bergman projection $B_t = I - \bar\partial^*_t N^t_{(p,1)} \bar\partial$ are proved following the same arguments as in Theorem 6.1.4 in [CS]. We will show that $\mathcal{H}_{(p,q)}(\Omega) = \{0\}$ using the following $\bar\partial$-closed extension of forms.

Theorem 3.2. *Let $\Omega \subset\subset \mathbb{C}^n$ be the same as in Theorem 3.1 with $n \geq 3$. For any $f \in L^2_{(p,q)}(\Omega)$, where $0 \leq p \leq n$ and $0 \leq q < n-1$, such that $\bar\partial f = 0$ in Ω, there exists $F \in W^{-1}_{(p,q)}(\Omega_1)$ such that $F|_\Omega = f$ and $\bar\partial F = 0$ in Ω_1 in the distribution sense.*

If $1 \leq q < n-1$, there exists $u \in L^2_{(p,q-1)}(\Omega)$ satisfying $\bar\partial u = f$ in Ω.

Proof. From Theorem 3.1, we have that smooth $\bar\partial$-closed forms are dense in $L^2_{(p,q)}(\Omega) \cap \mathrm{Ker}(\bar\partial)$ (see Corollary 6.1.6 in [CS]). We may assume that f is smooth and it suffices to prove *a priori* estimates. Let $\tilde f$ be the smooth extension of f so that $\tilde f|_\Omega = f$. The rest of the proof is similar to the proof of Theorem 2.2. We will only indicate the necessary changes. Let $v \in W^1_{(p,q+1)}(\Omega_2)$ and $\tilde v$ be an $W^1(\Omega_1)$ extension of v to Ω_1 and $\tilde v$ has compact support in Ω_1. Since f is $\bar\partial$-closed on Ω, we have

$$\begin{aligned} |(\bar\partial \tilde f, v)_{(\Omega_2)}| &= |(\bar\partial \tilde f, v)_{(\Omega_1)}| = |(\tilde f, \vartheta \tilde v)_{(\Omega_1)}| \\ &\leq \|\tilde f\|_{(\Omega_1)} \|\vartheta \tilde v\|_{(\Omega_1)} \leq C \|\tilde f\|_{(\Omega_1)} \|v\|_{W^1(\Omega_2)}. \end{aligned}$$

It follows that $\bar\partial \tilde f$ is in $W^{-1}(\Omega_2)$, defined as the dual of $W^1(\Omega_2)$ and

$$\|\bar\partial \tilde f\|_{W^{-1}_{(p,q+1)}(\Omega_2)} \leq C \|\tilde f\|_{L^2_{(p,q)}(\Omega_1)}.$$

Let $N^{\Omega_2}_{(n-p,n-q-1)}$ be the weighted $\bar\partial$-Neumann operator with weights $t|z|^2$ for some large t on Ω_2. Here we omit the dependence on t to avoid too many indices. It follows from Kohn [Ko2] that $N^{\Omega_2}_{(n-p,n-q-1)}$ is exact regular on $W^s(\Omega_2)$ for each $s > 0$ if we choose sufficiently large t. We define $T\tilde f$ by $T\tilde f = -\star_t\bar\partial N^{\Omega_2}_{(n-p,n-q-1)}(\star_{-t}\bar\partial\tilde f)$ on Ω_2, where $\star_t = \star e^{-t|z|^2}$ is the Hodge star operator adjusted to the weighted L^2 space. From Theorem 9.1.2 in [CS], $T\tilde f$ satisfies $\bar\partial T\tilde f = \bar\partial\tilde f$ in Ω_1 in the distribution sense if we extend $T\tilde f$ to be zero outside Ω_2. Furthermore, from the exact regularity of the weighted $\bar\partial$-Neumann operator and $\bar\partial N^{\Omega_2}_{(n-p,n-q-1)}$ in the dual of $W^1(\Omega_2)$, we have that

$$\|T\tilde f\|_{W^{-1}(\Omega_2)} \le C\|\bar\partial\tilde f\|_{W^{-1}_{(p,q+1)}(\Omega_2)} \le C\|\tilde f\|_{L^2_{(p,q)}(\Omega_1)}.$$

Since $T\tilde f$ is in the dual of $W^1(\Omega_2)$, the extension of $T\tilde f$ by zero to Ω_1 is continuous from $W^{-1}(\Omega_2)$ to $W^{-1}(\Omega_1)$. Define

$$F = \begin{cases} f, & x\in\overline{\Omega}, \\ \tilde f - T\tilde f, & x\in\Omega_2. \end{cases} \tag{3.2}$$

Then $F\in W^{-1}_{(p,q)}(\Omega_1)$ and F is a $\bar\partial$-closed extension of f.

It follows that $F = \bar\partial U$ for some $U\in W^{-1}_{(p,q-1)}(\Omega_1)$ where we can take U to be the canonical solution $\bar\partial^*_t N^{\Omega_1}_t F$ with respect to the weight $t|z|^2$ with large $t>0$. It follows that $U\in L^2(\Omega_1, \mathrm{loc})$ from the interior regularity.

If $1<q<n-1$, we can actually have that $U\in L^2(\Omega_1)$ from the boundary regularity for $\bar\partial\oplus\bar\partial^*_t$. Let ζ be a cut-off function which is supported in a tubular neighborhood V of $b\Omega_1$ such that $\zeta = 1$ in a neighborhood of $b\Omega_1$. We first show *a priori* estimates assuming U is in $L^2(\Omega_1)$. Then we have from the Hörmander's weighted estimates

$$\|\zeta U\|_{t(\Omega_1)} \le C(\|\bar\partial(\zeta U)\|_{t(\Omega_1)} + \|\bar\partial^*_t(\zeta U)\|_{t(\Omega_1)}) < \infty \tag{3.3}$$

since $\bar\partial U = f$ and $\bar\partial^*_t U = 0$ are in $L^2(\Omega\cap V)$ and $(\bar\partial\zeta)U$ is in $L^2(\Omega_1)$. The constant in (3.3) depends only on the diameter of Ω_1. To pass from *a priori* estimates to the real estimates, we approximate Ω_1 from inside by strongly pseudoconvex domains with smooth boundary. We refer the reader to the paper by Boas-Shaw [BS] or Michel-Shaw [MS] (see also the proof of Theorem 4.4.1 in [CS]) for details.

When $q=1$, we have to modify the solution. Let $F_1 = \bar\partial(\zeta U) = \bar\partial(\zeta)U + \zeta\bar\partial U\in L^2(\Omega_1)$. We write $F = \bar\partial(\zeta U) + \bar\partial((1-\zeta)U) = F_1 + F_2$. Let $U_1 = \bar\partial^* N^{\Omega_1}F_1$. Then $U_1\in L^2(\Omega_1)$ and $\bar\partial U_1 = F_1$. Since F_2 is a $\bar\partial$-closed form with compact support in Ω_1, we can solve $\bar\partial U_2 = F_2$ in $\mathbb{C}^n$ by convolution with the Bochner-Martinelli kernel (or solve the $\bar\partial$-equation on a large ball containing $\overline{\Omega}_1$). This gives that $U_2\in L^2(\mathbb{C}^n)$. Setting $u = U_1 + U_2$ and restricting u to Ω, we have $u\in L^2_{(p,0)}(\Omega)$ satisfying $\bar\partial u = f$ in Ω. Notice that the latter method can also be applied to the case when $1\le q<n-1$. $\square$

For $q = n-1$, there is an additional compatibility condition for the $\bar\partial$-closed extension of $(p, n-1)$-forms.

Theorem 3.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ between two pseudoconvex domains Ω_1 and Ω_2 with smooth boundary and $\Omega_2 \subset\subset \Omega_1$, $n \geq 2$. For any $\bar\partial$-closed $f \in L^2_{(p,n-1)}(\Omega)$, where $0 \leq p \leq n$, the following conditions are equivalent:*

(1) *There exists $F \in W^{-1}_{(p,n-1)}(\Omega_1)$ such that $F|_\Omega = f$ and $\bar\partial F = 0$ in Ω_1 in the distribution sense.*

(2) *The restriction of f to $b\Omega_2$ satisfies the compatibility condition*

$$\int_{b\Omega_2} f \wedge \phi = 0, \quad \phi \in W^1_{(n-p,0)}(\Omega_2) \cap Ker(\bar\partial). \tag{3.4}$$

(3) *There exists $u \in L^2_{(p,n-2)}(\Omega)$ satisfying $\bar\partial u = f$ in Ω.*

Proof. We remark that any h in $W^1(\Omega_2)$ has a trace in $W^{\frac{1}{2}}(b\Omega_2)$ and any $\bar\partial$-closed $(p, n-1)$-form with $L^2(\Omega)$ coefficients has a well-defined complex tangential trace in $W^{-\frac{1}{2}}(b\Omega_2)$ (see, e.g., [CS2]). Thus the pairing between f and ϕ in (2) is well defined.

Since the weighted $\bar\partial$-Neumann operator $N^2_{(p,1)}$ on Ω_2 with weights $t|z|^2$ is exact regular on $W^s(\Omega_2$ for sufficiently large t, the Bergman projection is bounded from $W^s_{(p,0)}(\Omega_2)$ to itself for any $s \geq 0$ (see Corollary 6.1.6 in [CS]). We have that the space $W^s_{(p,0)}(\Omega_2) \cap \mathrm{Ker}(\bar\partial)$ is dense in $L^2_{(p,0)}(\Omega_2) \cap \mathrm{Ker}(\bar\partial)$. In particular, $W^1_{(p,0)}(\Omega_2) \cap \mathrm{Ker}(\bar\partial)$ is infinite dimensional.

We first show that (3) implies (2). Since $\bar\partial f = 0$ and f is in $L^2(\Omega)$, the tangential part of f has a distribution trace $f_b \in W^{-\frac{1}{2}}(b\Omega)$ on the boundary. Let $u_\epsilon \to u$ and $\bar\partial u_\epsilon \to \bar\partial u = f$ in L^2 where $u_\epsilon \in C^1(\overline{\Omega})$. Let $\bar\partial_b$ be the tangential Cauchy-Riemann equations induced by restricting $\bar\partial$ to $b\Omega$. For any $h \in W^1_{(n-p,0)}(\Omega_2) \cap \mathrm{Ker}(\bar\partial)$, the restriction of h is in $W^{\frac{1}{2}}_{(n-p,0)}(b\Omega_2)$.

Let $f \in L^2_{(p,n-1)}(\Omega)$ be a $\bar\partial$-closed form with $\bar\partial f = 0$ in Ω. Suppose that f is $\bar\partial$-exact for some $u \in L^2_{(p,n-2)}(\Omega)$. Let ζ be a cut-off function with $\zeta = 1$ on $b\Omega_2$ and $\zeta = 0$ on $b\Omega_1$. Then f must satisfy the compatibility condition

$$\int_{b\Omega_2} f \wedge h = \int_{b\Omega} \bar\partial(\zeta u) \wedge h = \lim_\epsilon \int_{b\Omega} \bar\partial(\zeta u_\epsilon) \wedge h = \lim_\epsilon \int_{b\Omega} \zeta u_\epsilon \wedge \bar\partial h = 0.$$

This proves that (3) implies (2). To show that (2) implies (1), we will modify the arguments in the proof of Theorem 3.2.

Using the same notation as before, we first approximate f by smooth forms f_ν on Ω such that $f_\nu \to f$ and $\bar\partial f_\nu \to 0$ in $L^2(\Omega)$. Since $\bar\partial f_\nu$ is top degree, we can always solve $\bar\partial g_\nu = \bar\partial f_\nu$ with smooth g_ν and $g_\nu \to 0$. Thus we may assume that f can be approximated by smooth $\bar\partial$-closed forms f_ν in $L^2(\Omega)$ and denote the smooth extension of f_ν by $\tilde f_\nu$. We define $T\tilde f_\nu$ by $T\tilde f_\nu = -\star\bar\partial N^{\Omega_2}_{(n-p,0)}(\star\bar\partial\tilde f_\nu)$ on Ω_2.

From the proof of Theorem 9.1.3 in [CS], the space

$$\bar{\partial} T \tilde{f}_\nu = \bar{\partial} f_\nu - B^{\Omega_2}_{(n-p,0)}(\star \bar{\partial} \tilde{f}_\nu).$$

We claim that $B^{\Omega_2}_{(n-p,0)}(\star \bar{\partial} \tilde{f}_\nu) \to B^{\Omega_2}_{(n-p,0)}(\star \bar{\partial} \tilde{f}) = 0$. Let $\phi \in W^1_{(n-p,0)}(\Omega_2) \cap \mathrm{Ker}(\bar{\partial})$. Then from (2), we have

$$(\phi, \star \bar{\partial} \tilde{f}_\nu) = \int_{\Omega_2} \phi \wedge \bar{\partial} \tilde{f}_\nu = \int_{b\Omega_2} \phi \wedge f_\nu \to \int_{b\Omega_2} \phi \wedge f = 0.$$

From the regularity of the weighted $\bar{\partial}$-Neumann operator, we have $W^1_{(n-p,0)}(\Omega_2) \cap \mathrm{Ker}(\bar{\partial})$ is dense in $L^2_{(n-p,0)}(\Omega_2) \cap \mathrm{Ker}(\bar{\partial})$ (see Corollary 6.1.6 in [CS]). This gives that $\bar{\partial} T \tilde{f} = \bar{\partial} \tilde{f}$ in Ω_2 since $B^{\Omega_2}_{(n-p,0)}(\star \bar{\partial} \tilde{f}) = 0$. Furthermore, $\bar{\partial} T \tilde{f} = \bar{\partial} \tilde{f}$ in Ω_1 in the distribution sense if we extend $T\tilde{f}$ to be zero outside Ω_2.

Define F similarly as before,

$$F = \begin{cases} f, & x \in \overline{\Omega}, \\ \tilde{f} - T\tilde{f}, & x \in \Omega_2. \end{cases}$$

Then $F \in W^{-1}_{(p,q)}(\Omega_1)$ and F is a $\bar{\partial}$-closed extension of f. This proves that (2) implies (1).

To show that (1) implies (3), one can solve $F = \bar{\partial} U$ for some $U \in L^2_{(p,q-1)}(\Omega_1)$. Let $u = U$ on Ω, we have $u \in L^2_{(p,q-1)}(\Omega)$ satisfying $\bar{\partial} u = f$ in Ω. Thus (1) implies (3). □

Corollary 3.4. *Let Ω be the same as in Theorem* 3.3. *Then $\bar{\partial}$ has closed range in $L^2_{(p,n-1)}(\Omega)$ and the $\bar{\partial}$-Neumann operator $N_{(p,n-1)}$ exists on $L^2_{(p,n-1)}(\Omega)$. The space of harmonic $(p, n-1)$-forms $\mathcal{H}_{(p,n-1)}$ is of infinite dimension.*

Proof. That $\bar{\partial}$ has closed range follows from Condition (2) in Theorem 3.3. The Bergman space $W^1_{(n-p,0)}(\Omega_2)$ is infinite dimensional. Each will yield a $\bar{\partial}$-closed $(p, n-1)$-form F on Ω which is not $\bar{\partial}$-exact. Since the space $\mathcal{H}_{(p,n-1)}(\Omega)$ is isomorphic to the quotient space of L^2 $\bar{\partial}$-closed $(p, n-1)$-forms over the closed subspace space of $\bar{\partial}$-exact forms, we conclude that $\mathcal{H}_{(p,n-1)}(\Omega)$ is infinite dimensional. □

We summarize the closed range property and regularity for $\bar{\partial}$ on the annulus between two pseudoconvex domains in $\mathbb{C}^n$ in the following theorem.

Theorem 3.5. *Let $\Omega \subset\subset \mathbb{C}^n$ be the annulus domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$ between two pseudoconvex domains Ω_1 and Ω_2 with smooth boundary and $\Omega_2 \subset\subset \Omega_1$. Then the $\bar{\partial}$-Neumann operator $N_{(p,q)}$ exists on $L^2_{(p,q)}(\Omega)$ for $0 \le p \le n$ and $1 \le q \le n-1$. For any $f \in L^2_{(p,q)}(\Omega)$, we have*

$$f = \bar{\partial}\bar{\partial}^* N_{(p,q)} f + \bar{\partial}^* \bar{\partial} N_{(p,q)} f, \qquad 1 \le q \le n-2.$$

$$f = \bar{\partial}\bar{\partial}^* N_{(p,n-1)} f + \bar{\partial}^* \bar{\partial} N_{(p,n-1)} f + H_{(p,n-1)} f, \qquad q = n-1$$

where $H_{(p,n-1)}$ is the projection operator onto the harmonic space $\mathcal{H}_{(p,n-1)}(\Omega)$ which is infinite dimensional.

Suppose that $f \in W^s_{(p,q)}(\Omega)$, where $s \geq 0$ and $1 \leq q \leq n-1$. We assume that $\bar\partial f = 0$ in Ω for $q \leq n-1$ and if $q = n-1$, we assume furthermore that f satisfies the condition

$$\int_{b\Omega_2} f \wedge \phi = 0, \quad \phi \in W^1_{(n-p,0)}(\Omega_2) \cap Ker(\bar\partial).$$

Then there exists $u \in W^s_{(p,q)}(\Omega)$ satisfying $\bar\partial u = f$.

Proof. Since $\bar\partial$ has closed range in $L^2_{(p,q)}(\Omega)$ for all degrees, we have that the $\bar\partial$-Neumann operator exists (without weights). The proof is exactly the same as the proof of Theorem 4.4.1 in [CS]. The regularity for $\bar\partial$ follows from Theorem 3.1 for $q < n-1$ and the earlier work of [Sh1]. When $q = n-1$, we can trace the proof of Theorem 3.3 to see that there exists a $\bar\partial$-closed form $F \in W^{s-1}_{(p,q)}(\Omega_1)$ which is equal to f on Ω. Thus one can find a solution $u \in W^s_{(p,n-2)}(\Omega)$ satisfying $\bar\partial u = f$. □

Remark: All the results in this section can be extended to annulus between pseudoconvex domains in a Stein manifold with trivial modification.

4. The $\bar\partial$-equation on weakly pseudoconcave domains in $\mathbb{C}P^n$

Much of the results in Section 2 can be applied to the strongly pseudoconcave domains or complements of finite type pseudoconvex domains in $\mathbb{C}P^n$ without much change. For the $\bar\partial$-equation on a weakly pseudoconcave domain in $\mathbb{C}P^n$, we cannot use the weight function methods used in Section 3 since $\mathbb{C}P^n$ is not Stein. We have the following results obtained in the recent papers [CSW] and [CS2] for pseudoconcave domains in $\mathbb{C}P^n$ Lipschitz boundary. Related results for $\bar\partial$ on the pseudoconcave domains in $\mathbb{C}P^n$, see the paper by Henkin-Iordan [HI].

We recall that a domain is called Lipschitz if the boundary is locally the graph of a Lipschitz function. For some basic properties of Lipschitz domains, see the preliminaries in [Sh4].

Theorem 4.1. *Let Ω^+ be a pseudoconcave domain in $\mathbb{C}P^n$ with Lipschitz boundary, where $n \geq 3$. For any $f \in W^{1+\epsilon}_{(p,q)}(\Omega^+)$, where $0 \leq p \leq n$, $1 \leq q < n-1$, $p \neq q$ and $0 < \epsilon < \frac{1}{2}$, such that $\bar\partial f = 0$ in Ω^+, there exists $u \in W^{1+\epsilon}_{(p,q-1)}(\Omega^+)$ with $\bar\partial u = f$ in Ω^+. If $b\Omega^+$ is C^2, the statement is also true for $\epsilon = 0$.*

The proof of Theorem 4.1 depends on the $\bar\partial$-closed extension of forms, which in term depends on the following $\bar\partial$-Cauchy problem.

Proposition 4.2. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with Lipschitz boundary, $n \geq 3$. Suppose that $f \in L^2_{(p,q)}(\delta^{-t}, \Omega)$ for some $t > 0$, where $0 \leq p \leq n$ and $1 \leq q < n$. Assuming that $\bar\partial f = 0$ in $\mathbb{C}P^n$ with $f = 0$ outside Ω, then there exists $u_t \in L^2_{(p,q-1)}(\delta^{-t}, \Omega)$ with $u_t = 0$ outside Ω satisfying $\bar\partial u_t = f$ in the distribution sense in $\mathbb{C}P^n$. If $b\Omega$ is C^2, the statement is also true for $t = 0$.*

Proof. Following Takeuchi (see [Ta] or [CS1]), the weighted $\bar{\partial}$-Neumann operators N_t exists for forms in $L^2_{(n-p,n-q)}(\delta^t, \Omega)$. Let $\star_{(t)}$ denote the Hodge-star operator with respect to the weighted norm $L^2(\delta^t, \Omega)$. Then

$$\star_{(t)} = \delta^t \star = \star \delta^t$$

where $\star$ is the Hodge star operator with the unweighted L^2 norm. Since $f \in L^2_{(p,q)}(\delta^{-t}, \Omega)$, we have that $\star_{(-t)} f \in L^2_{(p,q)}(\delta^t, \Omega)$. Let u_t be defined by

$$u_t = - \star_{(t)} \bar{\partial} N_t \star_{(-t)} f. \tag{4.1}$$

Then $u_t \in L^2_{(p,q-1)}(\delta^{-t}, \Omega)$, since $\bar{\partial} N_t \star_{(-t)} f$ is in $\mathrm{Dom}(\bar{\partial}_t^*) \subset L^2_{(n-p,n-q+1)}(\delta^t, \Omega)$. Since $\bar{\partial}_t^* = \delta^{-t} \vartheta \delta^t = - \star_{(-t)} \bar{\partial} \star_{(t)}$, we have $\star_{(-t)} f \in \mathrm{Dom}(\bar{\partial}_t^*)$ and $\bar{\partial}_t^* \star_{(-t)} f = 0$ in Ω. This gives

$$\bar{\partial}_t^* N_t \star_{(-t)} f = N_t \bar{\partial}_t^* \star_{(-t)} f = 0. \tag{4.2}$$

From (3.2), we have $\bar{\partial} u_t = f$

First notice that $\star_{(-t)} (-1)^{p+q} \bar{\partial} N_t \star_{(-t)} f = \bar{\partial} N_t \star_{(-t)} f \in \mathrm{Dom}(\bar{\partial}_t^*)$. We also have $\bar{\partial}_t^* \star_{(-t)} u = (-1)^{p+q} \star_{(-t)} f$ in Ω. Extending u_t to be zero outside Ω, one can show that $\bar{\partial} u_t = f$ in $\mathbb{C}P^n$ using that the boundary is Lipschitz.

If $b\Omega^+$ is C^2, the $\bar{\partial}$-Neumann operator exists without weights. The above arguments can be applied to the case when $t = 0$. For details, see [CSW] and [CS2]. ☐

Proposition 4.3. *Let $\Omega \subset\subset \mathbb{C}P^n$ be a pseudoconvex domain with Lipschitz boundary and let $\Omega^+ = \mathbb{C}P^n \setminus \overline{\Omega}$. For any $f \in W^{1+\epsilon}_{(p,q)}(\Omega^+)$, where $0 \le p \le n$, $0 \le q < n-1$ and $0 < \epsilon < \frac{1}{2}$, such that $\bar{\partial} f = 0$ in Ω^+, there exists $F \in W^{\epsilon}_{(p,q)}(\mathbb{C}P^n)$ with $F|_{\Omega^+} = f$ and $\bar{\partial} F = 0$ in $\mathbb{C}P^n$ in the distribution sense.*

If $b\Omega$ is C^2, the statement is also true for $\epsilon = 0$.

Proof. Since Ω has Lipschitz boundary, there exists a bounded extension operator from $W^s(\Omega^+)$ to $W^s(\mathbb{C}P^n)$ for all $s \ge 0$ (see, e.g., [Gr]). Let $\tilde{f} \in W^{1+\epsilon}_{(p,q)}(\mathbb{C}P^n)$ be the extension of f so that $\tilde{f}|_{\Omega^+} = f$ with $\|\tilde{f}\|_{W^{1+\epsilon}(\mathbb{C}P^n)} \le C\|f\|_{W^{1+\epsilon}(\Omega^+)}$. Furthermore, we can choose an extension such that $\bar{\partial}\tilde{f} \in W^{\epsilon}(\Omega) \cap L^2(\delta^{-2\epsilon}, \Omega)$.

We define $T\tilde{f}$ by $T\tilde{f} = -\star_{(2\epsilon)} \bar{\partial} N_{2\epsilon} (\star_{(-2\epsilon)} \bar{\partial} \tilde{f})$ in Ω. From Proposition 4.2, we have that $T\tilde{f} \in L^2(\delta^{-2\epsilon}, \Omega)$. But for a Lipschitz domain, we have that $T\tilde{f} \in L^2(\delta^{-2\epsilon}, \Omega)$ is comparable to $W^{\epsilon}(\Omega)$ when $0 < \epsilon < \frac{1}{2}$. This gives that $T\tilde{f} \in W^{\epsilon}(\Omega)$ and $T\tilde{f}$ satisfies $\bar{\partial} T\tilde{f} = \bar{\partial}\tilde{f}$ in $\mathbb{C}P^n$ in the distribution sense if we extend $T\tilde{f}$ to be zero outside Ω.

Since $0 < \epsilon < \frac{1}{2}$, the extension by 0 outside Ω is a continuous operator from $W^{\epsilon}(\Omega)$ to $W^s(\mathbb{C}P^n)$ (see, e.g., [Gr]). Thus we have $T\tilde{f} \in W^{\epsilon}(\mathbb{C}P^n)$.

Define

$$F = \begin{cases} f, & x \in \overline{\Omega}^+, \\ \tilde{f} - T\tilde{f}, & x \in \Omega. \end{cases}$$

Then $F \in W^{\epsilon}_{(p,q)}(\mathbb{C}P^n)$ and F is a $\bar{\partial}$-closed extension of f. ☐

From Proposition 4.3, Theorem 4.1 follows easily.

A Lipschitz (or C^1) hypersurface is said to be Levi-flat if it is locally foliated by complex manifolds of complex dimension $n-1$. A C^2 hypersurface M is called Levi-flat if its Levi-form vanishes on M. Any C^k Levi-flat hypersurface, $k \geq 2$ is locally foliated by complex manifolds of complex dimension $n-1$. The foliation is of class C^k if the hypersurface is of class C^k, $k \geq 2$ (see Barrett-Fornaess [BF]). The proof in [BF] also gives that if a real C^1 hypersurface admits a continuous foliation by complex manifolds, then the foliation is actually C^1. One of the main applications of the $\bar{\partial}$-equation on pseudoconcave domains in $\mathbb{C}P^n$ is the following result (see [CS2]).

Theorem 4.4. *There exist no Lipschitz Levi-flat hypersurfaces in $\mathbb{C}P^n$ for $n \geq 3$.*

It was first proved in Lins-Neto [LNe] that there exist no real analytic Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$. Siu [Si] proved the nonexistence of smooth (or $\frac{3n}{2}+7$) Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$. The proof for the nonexistence of C^2 Levi-flat hypersurfaces in $\mathbb{C}P^2$ in [CSW] is still incomplete. The main missing ingredient is the lack of closed range property for $\bar{\partial}$ on pseudoconcave domains. In the case for an annulus domains in $\mathbb{C}^n$, notice that we have used the regularity of the weighted $\bar{\partial}$-Neumann operator on pseudoconvex domains proved by Kohn in the proof of Theorem 3.3. We end the section with the following three open questions.

Conjecture 1. *Let $\Omega^+ \subset\subset \mathbb{C}P^n$ be a pseudoconcave domain with C^2-smooth boundary (or Lipschitz) $b\Omega^+$, $n \geq 2$. Then $\bar{\partial} : L^2_{(p,q-1)}(\Omega)^+ \to L^2_{(p,q)}(\Omega)^+$ has closed range for $0 \leq p \leq n$ and $1 \leq q \leq n-1$.*

Conjecture 2. *Let Ω^+ be a C^2 pseudoconcave domain in $\mathbb{C}P^n$, $n \geq 2$. For any $0 \leq p \leq n$, the space of harmonic $(p, n-1)$-forms $\mathcal{H}_{(p,n-1)}$ is infinite dimensional and for any $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*) \cap (\mathcal{H}_{(p,n-1)})^\perp$, we have*

$$\|f\|^2 \leq C(\|\bar{\partial} f_1\|^2 + \|\bar{\partial}^* f_2\|^2).$$

Both Conjecture 1 and Conjecture 2 will imply the nonexistence of Levi-flat hypersurfaces in $\mathbb{C}P^2$.

Conjecture 3. *Let Ω be a pseudoconvex domain in $\mathbb{C}P^n$ with C^2 boundary, where $n \geq 2$. Then the range of $\bar{\partial}_b : L^2_{(p,q-1)}(b\Omega) \to L^2_{(p,q)}(b\Omega)$ is closed in the $L^2_{(p,q)}(b\Omega)$ space for all $0 \leq p \leq n$ and $1 \leq q \leq n-1$.*

When Ω is a smooth pseudoconvex domain in $\mathbb{C}^n$, Conjecture 3 is proved in Shaw [Sh2], Boas-Shaw [BS] and Kohn [Ko3] (see also Chapter 9 in Chen-Shaw [CS] and also Harrington [Ha] for C^1 pseudoconvex boundary). If Ω is Lipschitz pseudoconvex in $\mathbb{C}^n$ and we assume that there exists a plurisubharmonic defining function in a neighborhood of $\overline{\Omega}$, L^2 existence for $\bar{\partial}_b$ has been established in [Sh3].

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New Normal Forms for Levi-nondegenerate Hypersurfaces

Dmitri Zaitsev

Dedicated to Linda Preiss Rothschild on the occasion of her birthday

Abstract. In this paper we construct a large class of new normal forms for Levi-nondegenerate real hypersurfaces in complex spaces. We adopt a general approach illustrating why these normal forms are natural and which role is played by the celebrated Chern-Moser normal form. The latter appears in our class as the one with the "maximum normalization" in the lowest degree. However, there are other natural normal forms, even with normalization conditions for the terms of the same degree. Some of these forms do not involve the cube of the trace operator and, in that sense, are simplier than the one by Chern-Moser. We have attempted to give a complete and self-contained exposition (including proofs of well-known results about trace decompositions) that should be accessible to graduate students.

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1. Introduction

In this paper we construct a large class of new normal forms for *Levi-nondegenerate real hypersurfaces* in complex spaces. We adopt a general approach illustrating why these normal forms are natural and which role is played by the celebrated Chern-Moser normal form [CM74]. The latter appears in our class as the one with the *"maximum normalization" in the lowest degree*. However, there are other natural normal forms, even with normalization conditions for the terms of the same degree. Some of these forms do not involve the cube of the trace operator and, in that sense, are simpler than the one by Chern-Moser. We have attempted to give a complete and self-contained exposition (including proofs of well-known results about trace decompositions) that should be accessible to graduate students.

All normal forms here are formal, i.e., at the level of formal power series. This is sufficient for most purposes such as constructing invariants or solving the local equivalence problem for real-analytic hypersurfaces. In fact, a formal equivalence map between Levi-nondegenerate real-analytic hypersurfaces is automatically convergent and is therefore a local biholomorphic map. This is a special case of an important result of Baouendi-Ebenfelt-Rothschild [BER00a] and can also be obtained from the Chern-Moser theory [CM74]. The reader is referred for more details to an excellent survey [BER00b]. We also refer to normal forms for *Levi-degenerate hypersurfaces* [E98a, E98b, Ko05], for real CR submanifolds of *higher codimension* [ES95, SS03], for submanifolds with *CR singularities* [MW83, HY08], and for *non-integrable Levi-nondegenerate hypersurface type CR structures* [Z08].

Throughout the paper we consider a real-analytic hypersurface M in $\mathbb{C}^{n+1}$ passing through 0 and locally given by an equation

$$\mathsf{Im}\, w = \varphi(z, \bar{z}, \mathsf{Re}\, w), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}, \tag{1.1}$$

where we think of φ as a power series in the components of $z, \bar{z} \in \mathbb{C}^n$ and $u = \mathsf{Re}\, w \in \mathbb{R}$. If the given hypersurface is merely smooth, one can still consider the Taylor series of its defining equation, which is a formal power series. This motivates the notion of a *formal hypersurface*, i.e., the one given (in suitable coordinates) by (1.1) with φ being a formal power series.

In order to describe the normal forms, consider the expansion

$$\varphi(z, \bar{z}, u) = \sum_{k,l,m} \varphi_{kml}(z, \bar{z}) u^l, \tag{1.2}$$

where each $\varphi_{kml}(z, \bar{z})$ is a bihomogeneous polynomial in $(z, \bar{z})$ of bidegree (k, m), i.e., $\varphi_{kml}(tz, s\bar{z}) = t^k s^m \varphi_{kml}(z, \bar{z})$ for $t, s \in \mathbb{R}$. The property that φ is real is equivalent to the reality conditions

$$\overline{\varphi_{kml}(z, \bar{z})} = \varphi_{mkl}(z, \bar{z}). \tag{1.3}$$

The property that M passes through 0 corresponds to φ having *no constant terms.* Furthermore, after a complex-linear transformation of $\mathbb{C}^{n+1}$, one may assume that φ also has *no linear terms*, which will be our assumption from now on. The *Levi form* now corresponds to $\varphi_{11}(z, \bar{z})$, the only lowest order term that cannot be eliminated after a biholomorphic change of coordinates. Following [CM74], we write $\varphi_{11}(z, \bar{z}) = \langle z, z \rangle$, which is a hermitian form in view of (1.3). If the Levi form is nondegenerate, after a further complex-linear change of coordinates, we may assume that

$$\langle z, z \rangle = \sum_{j=1}^{n} \varepsilon_j z_j \bar{z}_j, \quad \varepsilon_j = \pm 1. \tag{1.4}$$

An important role in the normal forms is played by the *trace operator* associated with $\langle z, z \rangle$, which is the second-order differential operator given by

$$\mathsf{tr} := \sum_{j=1}^{n} \varepsilon_j \frac{\partial^2}{\partial z_j \partial \bar{z}_j}. \tag{1.5}$$

In particular, for $n = 1$ we have

$$\begin{aligned} \varphi_{kml}(z,\bar z) &= c_{kml} z^k \bar z^m, \\ \mathsf{tr}\varphi_{kml} &= \mathsf{tr}(c_{kml} z^k \bar z^m) = \begin{cases} kmc_{kml} z^{k-1}\bar z^{m-1}, & \min(k,m) \ge 1, \\ 0, & \min(k,m) = 0. \end{cases} \end{aligned} \tag{1.6}$$

We now consider the following normalization conditions.

1. For every $k \ge 2$ and $l \ge 1$, choose distinct integers $0 \le m, m' \le l$ with $m \ge 1$, $m' \ne m$ and consider the conditions
$$\mathsf{tr}^{m-1}\varphi_{k+m,m,l-m} = 0, \quad \mathsf{tr}^{m'}\varphi_{k+m',m',l-m'} = 0.$$
2. For every $k \ge 2$, consider the condition
$$\varphi_{k00} = 0.$$
3. For every $l \ge 2$, choose pairwise distinct integers $0 \le m, m', m'' \le l$ with $m \ge 1$, of which the nonzero ones are not of the same parity (i.e., neither all are even nor all are odd) and consider the conditions
$$\mathsf{tr}^{m-1}\varphi_{m+1,m,l-m} = 0, \quad \mathsf{tr}^{m'}\varphi_{m'+1,m',l-m'} = 0, \quad \mathsf{tr}^{m''}\varphi_{m''+1,m'',l-m''} = 0.$$
4. For every $l \ge 3$, choose distinct even integers $0 \le m, m' \le l$ with $m \ge 1$ and disjoint odd integers $0 \le \widetilde m, \widetilde m' \le l$ and consider the conditions
$$\begin{aligned} &\mathsf{tr}^{m-1}\varphi_{m,m,l-m} = 0, && \mathsf{tr}^{m'}\varphi_{m',m',l-m'} = 0, \\ &\mathsf{tr}^{\widetilde m-1}\varphi_{\widetilde m,\widetilde m,l-\widetilde m} = 0, && \mathsf{tr}^{\widetilde m'}\varphi_{\widetilde m',\widetilde m',l-\widetilde m'} = 0. \end{aligned}$$
5. Consider the conditions
$$\varphi_{101} = 0, \quad \varphi_{210} = 0, \quad \varphi_{002} = 0, \quad \varphi_{111} = 0, \quad \mathsf{tr}\varphi_{220} = 0.$$

In view of (1.5), in case $n = 1$ all traces can be omitted. The multi-indices (k, m, l) of φ_{kml} involved in different conditions (1)–(5) are all disjoint. They are located on parallel lines in the direction of the vector $(1, 1, -1)$. In fact, (1) corresponds to the lines through $(k + l, l, 0)$ with $k \ge 2$, $l \ge 1$, whereas (2) corresponds to the same lines with $l = 0$ containing only one triple $(k, 0, 0)$ with nonnegative components. Condition (3) corresponds to the lines through $(k + l, l, 0)$ for $k = 1$, $l \ge 2$, whereas (4) corresponds to the same lines for $k = 0$ and $l \ge 3$. Finally (5) involves all 5 coefficients that correspond to the lines through $(2, 1, 0)$ and $(2, 2, 0)$.

The following is the main result concerning the above normal forms.

Theorem 1.1. *Every (formal) real hypersurface M through 0 admits a local (formal) biholomorphic transformation h preserving 0 into each of the normal forms given by (1)–(5). If M is of the form (1.1) with φ having no constant and linear terms and satisfying (1.4), the corresponding transformation $h = (f, g)$ is unique provided it is normalized as follows:*

$$f_z = \mathsf{id}, \quad f_w = 0, \quad g_z = 0, \quad g_w = 1, \quad \mathsf{Re}\, g_{z^2} = 0, \tag{1.7}$$

where the subscripts denote the derivatives taken at the origin.

The Chern-Moser normal form corresponds to a choice of the coefficients of the lowest degree that may appear in (1)–(5). In fact, we choose $m = 1$, $m' = 0$ in (1), $m = 1$, $m' = 0$, $m'' = 2$ in (2), and $m = 2$, $m' = 0$, $\widetilde{m} = 1$, $\widetilde{m}' = 3$ in (4) and obtain the familiar Chern-Moser normal form

$$\varphi_{k0l} = 0, \quad \varphi_{k1l} = 0, \quad \mathsf{tr}\varphi_{22l} = 0, \quad \mathsf{tr}^2\varphi_{32l} = 0, \quad \mathsf{tr}^3\varphi_{33l} = 0, \tag{1.8}$$

where $(k, l) \neq (1, 0)$ in the second equation.

However, we also obtain other normal forms involving the same coefficients φ_{kml}. Namely, we can exchange $\widetilde{m}$ and $\widetilde{m}'$ in (4), i.e., choose $\widetilde{m} = 3$, $\widetilde{m}' = 1$, which leads to the normalization

$$\begin{array}{cccc} \varphi_{k0l} = 0, & \varphi_{k1l} = 0, & \mathsf{tr}\varphi_{11l} = 0, & \varphi_{111} = 0, \\ \mathsf{tr}\varphi_{22l} = 0, & \mathsf{tr}^2\varphi_{32l} = 0, & \mathsf{tr}^2\varphi_{33l} = 0, & \end{array} \tag{1.9}$$

where $k \geq 2$ in the second and $l \geq 2$ in the third equation. This normal form is simpler than (1.8) in the sense that it only involves the trace operator and its square rather than its cube. A comparison of (1.8) and (1.9) shows that the first set of conditions has more equations for φ_{11l}, whereas the second set has more equations for φ_{33l}. Thus we can say that the Chern-Moser normal form has the "maximum normalization" in the lowest degree.

Alternatively, we can exchange m and m'' in (2), i.e., choose $m = 2$, $m' = 0$, $m'' = 1$, which leads to the normalization

$$\begin{array}{cccc} \varphi_{k0l} = 0, & \varphi_{k1l} = 0, & \mathsf{tr}\varphi_{21l} = 0, & \varphi_{210} = 0, \\ \mathsf{tr}\varphi_{22l} = 0, & \mathsf{tr}\varphi_{32l} = 0, & \mathsf{tr}^3\varphi_{33l} = 0, & \end{array} \tag{1.10}$$

where $k = 1$, $l \geq 1$ or $k \geq 3$ in the second and $l \geq 1$ in the third equations. Again, comparing with (1.8), we see that the latter has more equations for φ_{21l} and less for φ_{32l}, i.e., again the Chern-Moser normal form has the "maximum normalization" in the lowest degree.

Finally, we can combine both changes leading to (1.9) and (1.10), i.e., choose $m = 2$, $m' = 0$, $m'' = 1$ in (2) and $\widetilde{m} = 3$, $\widetilde{m}' = 1$ in (4) and obtain yet another normal form involving the same terms φ_{kml}. We leave it to the reader to write the explicit equations.

Thus we have 4 normal forms involving the same terms as the one by Chern-Moser. We now describe a completely different normal form, which also has certain extremality property. Roughly speaking, in those 4 normal forms, we have chosen the multi-indices with the smallest first two components on each line of indices in (1), (3), (4). We now choose the multi-indices with the smallest last component, i.e., those at the *other end* of each line. For instance, we can choose $m = l$, $m' = l-1$ in (1), $m = l$, $m' = l - 1$, $m'' = l - 2$ in (3) and $m = l$, $m' = l - 2$, $\widetilde{m} = l - 1$, $\widetilde{m}' = l - 3$ or $\widetilde{m} = l$, $\widetilde{m}' = l - 2$, $m = l - 1$, $m' = l - 3$ in (4), depending on the parity of l. We obtain

$$\begin{array}{cccc} \mathsf{tr}^{l-1}\varphi_{kl0} = 0, & \varphi_{k00} = 0, & \mathsf{tr}^l\varphi_{kl1} = 0, & \mathsf{tr}^{l-1}\varphi_{ll1} = 0, \\ \mathsf{tr}^l\varphi_{l+1,l,2} = 0, & \mathsf{tr}^l\varphi_{ll2} = 0, & \mathsf{tr}^l\varphi_{ll3} = 0, & \end{array} \tag{1.11}$$

where $k \geq l \geq 1$, $(k,l) \neq (1,1)$ in the first, $k \geq l+1$ in the third, and $l \geq 1$ in the forth equations. This normal form is distinguished by the property that it has the "maximum normalization" of the terms φ_{kml} with the lowest index l. In fact, (1.11) only involves φ_{kml} with $l = 0,1,2,3$. In particular, most harmonic terms φ_{k0l} are not eliminated in contrast to the normal forms (1.8)–(1.10). In case $n = 2$ all traces can be removed, in particular, all terms φ_{kml} with $l = 0$, except the Levi form, are eliminated, i.e., $\varphi(z,\bar{z},u) = \langle z, z\rangle + O(|u|)$.

Finally, we consider another interesting normal form that in some sense mixes the one by Chern-Moser with the one in (1.11) (or, more precisely, the ones in (1.9) and (1.11)). Here the multi-indices on both ends of the lines are involved. This normal form corresponds to the choice of $m = l$, $m' = 0$ in (1), $m = l$, $m' = 0$, $m'' = l-1$ in (3) and $m = l$, $m' = 1$, $\widetilde{m} = l-1$, $\widetilde{m}' = 0$ if l is even and $\widetilde{m} = l$, $\widetilde{m}' = 1$, $m = l-1$, $m' = 0$ if l is odd. This leads to the normalization conditions

$$\varphi_{k0l} = 0, \quad \mathsf{tr}^{l-1}\varphi_{kl0} = 0, \quad \mathsf{tr}^{l}\varphi_{l+1,l,1} = 0, \quad \mathsf{tr}^{l-1}\varphi_{ll1} = 0, \quad \mathsf{tr}\varphi_{11l} = 0, \tag{1.12}$$

where $k \geq l \geq 1$, $(k,l) \neq (1,1)$ in the second, $l \geq 1$ in the forth, and $l \geq 2$ in the fifth equation. A remarkable feature of this normal form that distinguishes it from the previous ones, including the one by Chern-Moser, is that it only involves φ_{kml} with $\min(k,m,l) \leq 1$.

The rest of the paper is now devoted to the proof of Theorem 1.1, where we also explain how the normalization conditions (1)–(5) arise and why they are natural.

2. Transformation rule and its expansion

As before, we shall consider a real-analytic hypersurface M in $\mathbb{C}^{n+1}$ through 0, locally given by (1.1) with φ having no constant and linear terms. To the hypersurface M (or more precisely to the germ $(M,0)$) we apply a local biholomorphic transformation $(z,w) \mapsto (f(z,w), g(z,w))$ preserving 0 and transforming it into another germ $(M',0)$ of a real-analytic hypersurface in $\mathbb{C}^{n+1}$, still given by an equation

$$\mathsf{Im}\, w' = \varphi'(z', \bar{z}', \mathsf{Re}\, w'), \tag{2.1}$$

where φ' has no constant and linear terms. We consider (multi)homogeneous power series expansions

$$\begin{aligned} f(z,w) &= \sum f_{kl}(z) w^l, \quad g(z,w) = \sum g_{kl}(z) w^l, \\ \varphi(z,\bar{z},u) &= \sum \varphi_{kml}(z,\bar{z}) u^l, \quad \varphi'(z',\bar{z}',u') = \sum \varphi'_{kml}(z',\bar{z}') u'^l, \end{aligned} \tag{2.2}$$

where $f_{kl}(z)$ and $g_{kl}(z)$ are homogeneous polynomials in $z \in \mathbb{C}^n$ of degree k and $\varphi_{kml}(z,\bar{z})$ and $\varphi'_{kml}(z',\bar{z}')$ are bihomogeneous polynomials in $(z,\bar{z}) \in \mathbb{C}^n \times \mathbb{C}^n$ and $(z',\bar{z}') \in \mathbb{C}^n \times \mathbb{C}^n$ respectively of bidegree (k,l). Furthermore, f_{kl} and g_{kl} are abitrary whereas $\varphi_{kml}(z,\bar{z})$ and $\varphi'_{kml}(z',\bar{z}')$ satisfies the reality condition (1.3) which are equivalent to φ and φ' being real valued.

The fact that the map $(z,w) \mapsto (f(z,w), g(z,w))$ transforms $(M,0)$ into $(M',0)$ can be expressed by the equation

$$\begin{aligned} &\operatorname{Im} g(z, u+i\varphi(z,\bar z,u)) \\ &\quad = \varphi'\big(f(z,u+i\varphi(z,\bar z,u)), \overline{f(z,u+i\varphi(z,\bar z,u))}, \operatorname{Re} g(z,u+i\varphi(z,\bar z,u))\big). \end{aligned} \tag{2.3}$$

We use (2.2) to expand both sides of (2.3):

$$\operatorname{Im} g(z,u+i\varphi(z,\bar z,u)) = \operatorname{Im}\Big(\sum g_{kl}(z)\big(u+i\sum \varphi_{jhm}(z,\bar z)u^m\big)^l\Big), \tag{2.4}$$

$$\begin{aligned} &\varphi'\big(f(z,u+i\varphi(z,\bar z,u)), \overline{f(z,u+i\varphi(z,\bar z,u))}, \operatorname{Re} g(z,u+i\varphi(z,\bar z,u))\big) \\ &= \sum \varphi'_{klm}\Big(\sum f_{ab}(z)\big(u+i\sum\varphi_{jhr}(z,\bar z)u^r\big)^b, \overline{\sum f_{cd}(z)\big(u+i\sum\varphi_{jhr}(z,\bar z)u^r\big)^d}\Big) \\ &\qquad \times \Big(\operatorname{Re}\big(\sum g_{st}(z)\big(u+i\sum\varphi_{jhr}(z,\bar z)u^r\big)^t\big)\Big)^m. \end{aligned} \tag{2.5}$$

Since both φ and φ' have vanishing linear terms, collecting the linear terms in (2.4) and (2.5) and substituting into (2.3) we obtain

$$g_{10} = 0, \quad \operatorname{Im} g_{01} = 0. \tag{2.6}$$

Conditions (2.6) express the fact that the map (f,g) sends the normalized tangent space $T_0M = \mathbb{C}^n_z \times \mathbb{R}_u$ into the normalized tangent space $T_0M' = \mathbb{C}^n_{z'} \times \mathbb{R}_{u'}$ (where $u' = \operatorname{Re} w'$). The first condition in (2.6) implies that the Jacobian matrix of (f,g) at 0 is block-triangular and hence its invertibility is equivalent to the invertibility of both diagonal blocks $f_{10} = f_z(0)$ and $g_{01} = g_w(0)$.

3. Partial normalization in general case

Our first goal is to obtain a general normalization procedure that works for all series φ without any nondegeneracy assumption. The key starting point consists of identifying non-vanishing factors in (2.4) and (2.5). These are f_{10} and g_{01}. All other factors may vanish. Then we look for terms in the expansions of (2.4) and (2.5) involving at most one factor that may vanish. These are

$$\operatorname{Im} g_{kl}(z)u^l, \quad g_{01}\varphi_{kml}(z,\bar z)u^l, \quad \varphi'_{kml}(f_{10}(z), \overline{f_{10}(z)})(g_{01}u)^l, \tag{3.1}$$

where we used the reality of g_{01} and (1.3) and have dropped the argument z for g_{01} since the latter is a constant. The first term in (3.1) has only g_{kl} which may vanish, the second φ_{klm} and the third φ'_{klm}. Other summands have more than one entry (term) that may vanish. The terms (3.1) play a crucial role in the normalization and are called here the "good" terms. The other terms are called the "bad" terms. Good terms can be used to obtain a partial normalization of M as follows.

Consider in (2.3) the terms of multi-degree $(k,0,l)$ in $(z,\bar{z},u)$, where the first term from (3.1) appears. We obtain

$$\begin{aligned}\frac{1}{2i}g_{kl}(z)u^l + g_{01}\varphi_{k0l}(z,\bar{z})u^l &= \varphi'_{k0l}(f_{10}(z),\overline{f_{10}(z)})(g_{01}u)^l + \cdots, \quad k>0,\\ \mathsf{Im}\, g_{0l}u^l + g_{01}\varphi_{00l}u^l &= \varphi'_{00l}(g_{01}u)^l + \cdots,\end{aligned} \tag{3.2}$$

where the dots stand for the "bad" terms. If no dots were present, one can suitably choose g_{kl} in the first equation and $\mathsf{Im}\, g_{0l}$ in the second to obtain the normalization

$$\varphi'_{k0l}(z) = 0. \tag{3.3}$$

In presence of the "bad" terms, an induction argument is used. In fact, an inspection of the expansions of (2.4) and (2.5) shows that the terms in (3.2) included in the dots in (3.2), involve other coefficients $g_{st}(z)$ only of order $s+t$ less than $k+l$. Thus the normalization (3.3) can be obtained by induction on the order $k+l$. Furthermore, the expansion terms $g_{kl}(z)$ for $k>0$ and $\mathsf{Im}\, g_{0l}$ are uniquely determined by this normalization. However, the infinitely many terms $f_{kl}(z)$ and the real parts $\mathsf{Re}\, g_{0l}$ are not determined and act as free parameters. For every choice of those parameters, one obtains a germ $(M',0)$ which is biholomorphically equivalent to $(M,0)$ and satisfies the normalization (3.3). Thus this normalization is "partial". Normalization (3.3) is the well-known elimination of the so-called *harmonic terms* and works along the same lines also when M is of higher codimension, i.e., when w is a vector.

4. Levi-nondegenerate case

The good terms in (3.1) were not enough to determine $f_{kl}(z)$ and $\mathsf{Re}\, g_{0l}$. Thus we need more good terms for a complete normal form. These new "good" terms must be obtained from the expansion of (2.5) since $f_{kl}(z)$ do not appear in (2.4). Thus we need a nonvanishing property for some φ'_{kml}. The well-known lowest-order invariant of $(M,0)$ is the Levi form $\varphi'_{110}(z,\bar{z})$ which we shall write following [CM74] as $\langle z,z\rangle$. In view of (1.3), $\langle z,z\rangle$ is a hermitian form. We assume it to be complex-linear in the first and complex-antilinear in the second argument. The basic assumption is now that the Levi form is *nondegenerate*.

Once the class of all M is restricted to Levi-nondgenerate ones, we obtain further "good" terms involving the new nonvanishing factor $\langle z,z\rangle$. The good terms in both (2.5) and (2.4) come now from the expansion of

$$\begin{aligned}&\mathsf{Im}\,\big(g_{kl}(z)(u+i\langle z,z\rangle)^l\big), \quad 2\mathsf{Re}\,\big(\langle f_{kl}(z),f_{10}(z)\rangle(u+i\langle z,z\rangle)^l\big),\\ &g_{01}\varphi_{kml}(z,\bar{z})u^l, \quad \varphi'_{kml}(f_{10}(z),\overline{f_{10}(z)})(g_{01}u)^l.\end{aligned} \tag{4.1}$$

This time each of the coefficients g_{kl} and f_{kl} appears in some good term and thus can be potentially uniquely determined. The linear coefficients g_{01} and $f_{10}(z)$ play a special role. They appear by themselves in the good terms $\mathsf{Im}\, g_{01}$, $\mathsf{Re}\, g_{01}\langle z,z\rangle$

and $\langle f_{10}(z), f_{10}(z)\rangle$ in the expansions of (2.3) in multidegrees $(0,0,1)$ and $(1,1,0)$. Equating terms of those multidegrees, we obtain

$$\operatorname{Im} g_{01} = 0, \quad \operatorname{Re} g_{01}\langle z, z\rangle = \langle f_{10}(z), f_{10}(z)\rangle, \tag{4.2}$$

where the first condition already appeared in (2.6) and the second expresses the invariance of the Levi form (i.e., its transformation rule as tensor). The restrictions (4.2) describe all possible values of g_{01} and $f_{10}(z)$, which form precisely the group G_0 of all linear automorphism of the hyperquadric $\operatorname{Im} w = \langle z, z\rangle$. In order to study the action by more general biholomorphic maps $(z,w) \mapsto (f(z,w), g(z,w))$ satisfying (4.2), it is convenient to write a general map as a composition of one from G_0 and one satisfying

$$g_{01} = 1, \quad f_{10}(z) = z, \tag{4.3}$$

and study their actions separately. Since the action by the linear group G_0 is easy, we shall in the sequel consider maps (f, g) satisfying (4.3), unless specified otherwise.

Since the Levi form $\langle z, z\rangle$ is of bidegree $(1,1)$ in $(z, \bar z)$, we conclude from the binomial expansion of the powers in the first line of (4.1) that every coefficient g_{kl} contributes to the good terms in multidegrees $(k+m, m, l-m)$ and $(m, k+m, l-m)$ in $(z, \bar z, u)$ for all possible $0 \le m \le l$. The latters are the integral points with nonnegative components of the lines passing through the points $(k, 0, l)$ and $(0, k, l)$ in the direction $(1, 1, -1)$. Similarly, every coefficient f_{kl} contributes to the multidegrees $(k+m, m+1, l-m)$ and $(m+1, k+m, l-m)$ for all possible $0 \le m \le l$, corresponding to the lines passing through $(k, 1, l)$ and $(1, k, l)$ in the same direction $(1, 1, -1)$. For convenience, we shall allow one or both of k, l being negative, in which case the corresponding terms are assumed to be zero. We see that good terms with g_{kl} for $k > 0$ appear in two lines, whereas for g_{0l} both lines coincide with that through $(0, 0, l)$. Similarly, each f_{kl} with $k > 1$ or $k = 0$ appears in two lines, whereas for f_{1l} both lines coincide with that through $(1, 1, l)$. Thus the lines through $(0, 0, l)$ are special as well as the lines through $(1, 0, l)$ and $(0, 1, l)$ next to it. The latter lines contain good terms with f_{0l}, $f_{2,l-1}$ and g_{1l}. Each other line contains good terms with precisely one f_{kl} and one g_{kl}.

Thus we treat those groups of lines separately. Collecting in (2.3) terms of multi-degree $(k+m, m, l-m)$ in $(z, \bar z, u)$ for $k \ge 2$ we obtain

$$\begin{aligned} &\frac{1}{2i}\binom{l}{m} g_{kl}(z) u^{l-m} (i\langle z, z\rangle)^m + \varphi_{k+m,m,l-m}(z, \bar z) u^{l-m} \\ &= \binom{l-1}{m-1} \langle f_{k+1,l-1}(z), z\rangle u^{l-m} (i\langle z, z\rangle)^{m-1} + \varphi'_{k+m,m,l-m}(z, \bar z) u^{l-m} + \cdots , \end{aligned} \tag{4.4}$$

where as before the dots stand for all bad terms. Note that due to our convention, for $m = 0$, the term with $\binom{l-1}{m-1} = 0$ is not present. Similarly we collect terms of multi-degree $(m+1, m, l-m)$, this time we obtain two different terms with f_{ab}:

$$\frac{1}{2i}\binom{l}{m} g_{1l}(z)u^{l-m}(i\langle z,z\rangle)^m + \varphi_{m+1,m,l-m}(z,\bar z)u^{l-m}$$
$$= \binom{l-1}{m-1}\langle f_{2,l-1}(z), z\rangle u^{l-m}(i\langle z,z\rangle)^{m-1} + \binom{l-1}{m-1}\langle z, f_{0l}\rangle u^{l-m}(-i\langle z,z\rangle)^m$$
$$+ \varphi'_{m+1,m,l-m}(z,\bar z)u^{l-m} + \cdots, \quad (4.5)$$

where we have dropped the argument z from f_{0l} since the latter is a constant. Finally, for the terms of multidegree $(m, m, l-m)$, we have

$$\binom{l}{m}\mathsf{Im}\left(g_{0l}u^{l-m}(i\langle z,z\rangle)^m\right) + \varphi_{m,m,l-m}(z,\bar z)u^{l-m}$$
$$= 2\binom{l-1}{m-1}\mathsf{Re}\left(\langle f_{1,l-1}(z), z\rangle u^{l-m}(i\langle z,z\rangle)^{m-1}\right) + \varphi'_{m,m,l-m}(z,\bar z)u^{l-m} + \cdots.$$
$$(4.6)$$

Since both sides of (2.3) are real, its multihomogeneous part of a multi-degree (a, b, c) is conjugate to that of multi-degree (b, a, c). Hence the system of all equations in (4.4)–(4.6) is equivalent to (2.3), i.e., to the property that the map $(z, w) \mapsto (f(z, w), g(z, w))$ sends $(M, 0)$ into $(M', 0)$.

5. Weight estimates

In order to set up an induction similar to that of Section 3, we have to estimate the degrees of f_{kl} and g_{kl} appearing in the bad terms in (4.4)–(4.6) and compare it with the degrees of the good terms. However, different good terms with g_{kl} (see (4.1)) do not have the same degree but rather have the same *weight* $k+2l$, where the weight of z and $\bar z$ is 1 and the weight of u is 2. Similarly, different good terms with f_{kl} have the same weight $k+2l+1$. Hence this weight is more suitable to separate good terms from bad ones. Thus, instead of proceeding by induction on the degree as in Section 3, we proceed by induction on the weight.

We first inspect the weights of the bad terms in the expansions of (2.4) and (2.5). Recall that both φ and φ' have no constant or linear terms. Hence the weight of $\varphi_{jhm}(z,\bar z)u^m$ is greater than 2 unless $(j, h, m) \in \{(1, 1, 0), (2, 0, 0), (0, 2, 0)\}$. In particular, the expansion of

$$g_{kl}(z)(u + i\langle z,z\rangle + i\varphi_{200}(z,\bar z) + i\varphi_{020}(z,\bar z))^l$$

contains bad terms of the same weight $k+2l$ as "good" terms. The latter fact is not suitable for setting up the induction on the weight. This problem is solved by initial prenormalization of $(M, 0)$ as follows. According to Section 3, one can always eliminate harmonic terms from the expansion of φ. For our purposes, it will suffice to eliminate φ_{200} (and hence φ_{020} in view of (1.3)). Thus in the sequel, we shall assume that $\varphi_{200}, \varphi_{020}, \varphi'_{200}, \varphi'_{020}$ are all zero.

With that assumption in mind, coming back to (2.4), we see that the weight of $\varphi_{jhm}(z,\bar z)u^m$ is always greater than 2 unless we have $(j,h,m)=(1,1,0)$, that is $\varphi_{jhm}(z,\bar z)u^m = \langle z,z\rangle$. Since any bad term in the expansion of $g_{kl}(z)\big(u + i\sum\varphi_{jhm}(z,\bar z)u^m\big)^l$ contains at least one factor $\varphi_{jhm}(z,\bar z)u^m$ with $(j,h,m)\neq(1,1,0)$, its weight is greater than $k+2l$. Since the weights of all terms (including bad ones) in (4.4)–(4.6) are $k+2l$, the bad terms there coming from (2.4) can only contain $g_{ab}(z)$ or $\overline{g_{ab}(z)}$ with weight $a+2b<k+2l$, which is now suitable for our induction. Inspecting now the terms with $g_{st}(z)$ and $\overline{g_{st}(z)}$ in the expansion of (2.5), we see that their weights must be greater than $s+2t$. Hence, the bad terms in (4.4)–(4.6) coming from (2.5) can only contain g_{ab} or its conjugate with weight $a+2b<k+2l$.

Similarly, we inspect "bad" terms containing f_{ab}. This time we only need to look at the expansion of (2.5). A term in the expansion of

$$f_{ab}(z)\big(u+i\sum\varphi_{jhm}(z,\bar z)u^m\big)^b \tag{5.1}$$

is of weight greater than $a+2b$ unless it appears in the expansion of

$$f_{ab}(z)\big(u+i\langle z,\bar z\rangle\big)^b. \tag{5.2}$$

Keeping in mind that φ' has no linear terms, we conclude that a bad term in the expansion of (2.5) containing f_{ab} is always of weight greater than $a+2b+1$. The same holds for bad terms containing $\overline{f_{ab}}$. Hence, a bad term in (4.4)–(4.6) can only contain f_{ab} or its conjugate with weight $a+2b<k+2l-1$.

Summarizing, we obtain that bad terms in (4.4)–(4.6) can only contain g_{ab} or its conjugate with weight $a+2b<k+2l$ and f_{ab} or its conjugate with weight $a+2b<k+2l-1$. On the other hand, the good terms contain g_{ab} (or $\mathsf{Im}\, g_{ab}$) of weight precisely $k+2l$ and f_{ab} of weight precisely $2k+l-1$. Thus we may assume by induction on the weight that all terms denoted by dots in (4.4)–(4.6) are fixed and proceed by normalizing the good terms there.

6. Trace decompositions

The good terms involving $g_{kl}(z)$ appear as products of the latters and a power of the Levi form $\langle z,z\rangle$. As $g_{kl}(z)$ varies, these products

$$g_{kl}(z)\langle z,z\rangle^s \tag{6.1}$$

form a vector subspace of the space of all bihomogeneous polynomials in $(z,\bar z)$ of the corresponding bidigree $(k+s,s)$. In order to normalize such a product, we need to construct a complementary space to the space of all products (6.1). The latter is done by using the so-called trace decompositions described as follows.

Since the Levi form $\langle z,z\rangle$ is assumed to be nondegenerate, we can choose coordinates $z=(z_1,\dots,z_n)$ such that (1.4) is satisfied and consider the associated trace operator (1.5). The rest of this section is devoted to the proof of the following well-known trace decompositions (see [F17, S89, ES95] for more general decomposition results):

Proposition 6.1. *For every polynomial $P(z,\bar z)$, there exist unique polynomials $Q(z,\bar z)$ and $R(z,\bar z)$ such that*

$$P(z,\bar z) = Q(z,\bar z)\langle z,z\rangle^s + R(z,\bar z), \quad \mathsf{tr}^s R = 0. \tag{6.2}$$

Taking bihomogeneous components of all terms in (6.2) and using the uniqueness we obtain:

Lemma 6.2. *If P in Proposition* 6.1 *is bihomogeneous in $(z,\bar z)$, so are Q and R.*

Since the trace operator is real (maps real functions into real ones), we can take real parts of both sides in (6.2) and use the uniqueness to obtain:

Lemma 6.3. *If P in Proposition* 6.1 *is real, so are Q and R.*

We begin the proof of Proposition 6.1 with the following elementary lemma.

Lemma 6.4. *Let $P(z,\bar z)$ be a bihomogeneous polynomial of bidegree (p,q). Then*

$$\sum_j P_{z_j}(z,\bar z)z_j = pP(z,\bar z), \quad \sum_j P_{\bar z_j}(z,\bar z)\bar z_j = qP(z,\bar z), \tag{6.3}$$

Proof. By the assumption, $P(sz,t\bar z) = s^p t^q P(z,\bar z)$. Differentiating in s for $s=t=1$ we obtain the first identity in (6.3). Similarly, differentiating in t for $s=t=1$ we obtain the second identity. □

The following is the key lemma in the proof of Proposition 6.1.

Lemma 6.5. *Let $P(z,\bar z)$ be a bihomogeneous polynomial of bidegree (p,q). Then*

$$\mathsf{tr}\big(P(z,\bar z)\langle z,z\rangle\big) = (n+p+q)P(z,\bar z) + \big(\mathsf{tr}P(z,\bar z)\big)\langle z,z\rangle. \tag{6.4}$$

Proof. By straightforward calculations, we have

$$\begin{aligned}
\mathsf{tr}\big(P(z,\bar z)\langle z,z\rangle\big) &= \sum_j \varepsilon_j \frac{\partial^2}{\partial z_j \partial \bar z_j}\Big(P(z,\bar z)\sum_s \varepsilon_s z_s \bar z_s\Big) \\
&= \Big(\sum_j \varepsilon_j \frac{\partial^2}{\partial z_j \partial \bar z_j} P(z,\bar z)\Big)\sum_s \varepsilon_s z_s \bar z_s + \sum_j \varepsilon_j P_{z_j}(z,\bar z)\varepsilon_j z_j \\
&\quad + \sum_j \varepsilon_j P_{\bar z_j}(z,\bar z)\varepsilon_j \bar z_j + P(z,\bar z)\sum_j \varepsilon_j^2.
\end{aligned} \tag{6.5}$$

Using (6.3) we obtain the right-hand side of (6.4) as desired. □

Proof of Proposition 6.1 *for $s=1$.* We begin by proving the uniqueness of the decomposition (6.2). Let $P(z,\bar z)$ be bihomogeneous in $(z,\bar z)$ of bidegree (p,q) and suppose that

$$P(z,\bar z) = Q(z,\bar z)\langle z,z\rangle + R(z,\bar z), \quad \mathsf{tr}R = 0. \tag{6.6}$$

Applying $k \geq 1$ times tr to both sides of (6.6) and using (6.4) we obtain, by induction on k,

$$\mathsf{tr}^k P(z,\bar z) = c_k \mathsf{tr}^{k-1} Q(z,\bar z) + \big(\mathsf{tr}^k Q(z,\bar z)\big)\langle z,z\rangle, \tag{6.7}$$

where c_k are positive integers depending only on n, p, q and satisfying

$$c_1 = n + p + q - 2, \quad c_{k+1} = c_k + n + p + q - 2k - 2. \tag{6.8}$$

Since applying tr decreases both degrees in z and $\bar{z}$ by 1, one has $\mathsf{tr}^{k_0} Q = 0$ for $k_0 := \min(p, q)$. Hence $\mathsf{tr}^{k_0-1} Q$ is uniquely determined from (6.7) for $k = k_0$. Then going backwards through the identities (6.7) for $k = k_0 - 1, k_0 - 2, \ldots, 1$, we see that each $\mathsf{tr}^{k-1} Q$ is uniquely determined including $\mathsf{tr}^0 Q = Q$. Then R is uniquely determined by (6.6) proving the uniqueness part of Proposition 6.1 for $s = 1$.

To prove the existence, consider the equations

$$\mathsf{tr}^k P(z, \bar{z}) = c_k Q_{k-1}(z, \bar{z}) + Q_k(z, \bar{z})\langle z, z\rangle, \quad k = 1, \ldots, k_0, \tag{6.9}$$

obtained from (6.7) by replacing each $\mathsf{tr}^k Q(z, \bar{z})$ with an indeterminant polynomial $Q_k(z, \bar{z})$, where $Q_{k_0} = 0$ for the bidegree reason. Hence the last equation for $k = k_0$ reads $\mathsf{tr}^{k_0} P = c_{k_0} Q_{k_0-1}$, which we can solve for Q_{k_0-1}. Going backwards through the equations (6.9) for $k = k_0 - 1, k_0 - 2, \ldots, 1$, as before, we can solve the system (6.9) uniquely for $Q_{k_0-2}, \ldots, Q_0$. We claim that

$$Q_k = \mathsf{tr} Q_{k-1}, \quad k = 1, \ldots, k_0. \tag{6.10}$$

Indeed, (6.10) clearly holds for $k = k_0$ for bidegree reason. Suppose (6.10) holds for $k > k_1$. Applying tr to both sides of (6.9) for $k = k_1$, and using (6.4) we obtain

$$\mathsf{tr}^{k_1+1} P(z, \bar{z}) = c_{k_1} \mathsf{tr} Q_{k_1-1}(z, \bar{z}) + (n+p+q-2k-2) Q_{k_1}(z, \bar{z}) + \big(\mathsf{tr} Q_{k_1}(z, \bar{z})\big)\langle z, z\rangle, \tag{6.11}$$

which we compare to (6.9) for $k = k_1 + 1$:

$$\mathsf{tr}^{k_1+1} P(z, \bar{z}) = c_{k_1+1} Q_{k_1}(z, \bar{z}) + \big(\mathsf{tr} Q_{k_1}(z, \bar{z})\big)\langle z, z\rangle, \tag{6.12}$$

where we have used (6.10) for $k = k_1 + 1$. Using (6.8), we immediately obtain (6.10) for $k = k_1$. Thus (6.10) holds for all k by induction. In particular, substituting into (6.9) for $k = 1$, we obtain

$$\mathsf{tr} P(z, \bar{z}) = c_1 Q_0(z, \bar{z}) + \big(\mathsf{tr} Q_0(z, \bar{z})\big)\langle z, z\rangle = \mathsf{tr}\big(Q_0(z, \bar{z})\langle z, z\rangle\big), \tag{6.13}$$

where we have used (6.4) and (6.8). Thus we can take $Q(z, \bar{z}) := Q_0(z, \bar{z})$ and $R(z, \bar{z}) := P(z, \bar{z}) - Q_0(z, \bar{z})\langle z, z\rangle$ to satisfy (6.6), proving the existence part of Proposition 6.1 for $s = 1$. □

In Proposition 6.1 in general case we shall use the following lemma.

Lemma 6.6. *Assume that a polynomial* $P(z, \bar{z})$ *satisfies* $\mathsf{tr} P = 0$. *Then*

$$\mathsf{tr}^s\big(P(z, \bar{z})\langle z, z\rangle^{s-1}\big) = 0 \tag{6.14}$$

for any $s \geq 1$.

Proof. Using (6.4) for $P(z, \bar{z})$ replaced with $P(z, \bar{z})\langle z, z\rangle^{s-1}$ we obtain

$$\begin{aligned} \mathsf{tr}\big(P(z, \bar{z})\langle z, z\rangle^s\big) &= \mathsf{tr}\big(P(z, \bar{z})\langle z, z\rangle^{s-1}\langle z, z\rangle\big) \\ &= c_s P(z, \bar{z})\langle z, z\rangle^{s-1} + \big(\mathsf{tr}\big(P(z, \bar{z})\langle z, z\rangle^{s-1}\big)\big)\langle z, z\rangle \end{aligned} \tag{6.15}$$

for suitable integer c_s depending on s. Replacing $\mathsf{tr}\big(P(z,\bar z)\langle z,z\rangle^{s-1}\big)$ with the right-hand side of (6.15) for s replaced with $s-1$ and continuing the process we obtain

$$\mathsf{tr}\big(P(z,\bar z)\langle z,z\rangle^{s}\big) = c'_s P(z,\bar z)\langle z,z\rangle^{s-1}, \tag{6.16}$$

where c'_s is another integer depending on s. Now (6.14) can be proved directly by induction on s using (6.16). □

Proof of Proposition 6.1 *in the general case.* We prove the statement by induction on s. Since it has been proved for $s=1$, it remains to prove the induction step. Suppose for given P we have a decomposition (6.2). Applying Proposition 6.1 for $s=1$ to Q, we also obtain

$$Q(z,\bar z) = Q'(z,\bar z)\langle z,z\rangle + R'(z,\bar z), \quad \mathsf{tr}R' = 0. \tag{6.17}$$

Substituting into (6.2), we obtain

$$P(z,\bar z) = Q'(z,\bar z)\langle z,z\rangle^{s+1} + R'(z,\bar z)\langle z,z\rangle^{s} + R(z,\bar z), \quad \mathsf{tr}^s R = 0, \quad \mathsf{tr}R' = 0. \tag{6.18}$$

Furthermore, $\mathsf{tr}^{s+1}\big(R'(z,\bar z)\langle z,z\rangle^s\big) = 0$ by Lemma 6.6 and therefore we obtain a decomposition

$$P(z,\bar z) = Q'(z,\bar z)\langle z,z\rangle^{s+1} + R''(z,\bar z), \quad \mathsf{tr}^{s+1}R'' = 0, \tag{6.19}$$

as desired with $R''(z,\bar z) := R'(z,\bar z)\langle z,z\rangle^s + R(z,\bar z)$. This proves the existence part.

Clearly it suffices to prove the uniqueness for $P=0$. Assume it holds for s and that there is another decomposition

$$0 = \widetilde Q(z,\bar z)\langle z,z\rangle^{s+1} + \widetilde R(z,\bar z), \quad \mathsf{tr}^{s+1}\widetilde R = 0. \tag{6.20}$$

By Proposition 6.1 for $s=1$, we can write

$$\widetilde R = \widetilde Q'(z,\bar z)\langle z,z\rangle + \widetilde R'(z,\bar z), \quad \mathsf{tr}\widetilde R' = 0. \tag{6.21}$$

Substitution into (6.20) yields

$$0 = \big(\widetilde Q(z,\bar z)\langle z,z\rangle^s + \widetilde Q'(z,\bar z)\big)\langle z,z\rangle + \widetilde R'(z,\bar z), \quad \mathsf{tr}\widetilde R' = 0. \tag{6.22}$$

Then the uniqueness for $s=1$ implies $\widetilde R' = 0$ and

$$\widetilde Q(z,\bar z)\langle z,z\rangle^s + \widetilde Q'(z,\bar z) = 0. \tag{6.23}$$

Applying tr^{s+1} to both sides of (6.21) we obtain

$$\mathsf{tr}^{s+1}\big(\widetilde Q'(z,\bar z)\langle z,z\rangle\big) = 0. \tag{6.24}$$

Now using the identities (6.7) for $P(z,\bar z) := \widetilde Q'(z,\bar z)\langle z,z\rangle$, Q replaced by $\widetilde Q'$ and $k = s+1,\dots,k_0$ with k_0 as chosen there, we conclude that $\mathsf{tr}^s\widetilde Q' = 0$. Then the uniqueness in (6.23) implies $\widetilde Q = 0$. Hence $\widetilde R = 0$ by (6.20) and the proof is complete. □

7. Normalizations

We now proceed with normalization of the equation for M', i.e., of the terms $\varphi'_{klm}(z,\bar z)$. As noted at the end of §5, we may assume by induction on the weight that all "bad" terms denoted by dots in (4.4)–(4.6) are fixed. As explained in §5, every coefficient g_{kl} (resp. f_{kl}) contributes to good terms corresponding to the lines through $(k,0,l)$, $(0,k,l)$ (resp. $(k,1,l)$, $(1,k,l)$) in the same direction $(1,1,-1)$.

7.1. Normalization for $k \geq 2$

We begin by analysing the line through $(k,0,l)$ for $k \geq 2$, corresponding to the multi-degrees $(k+m,m,l-m)$, $0 \leq m \leq l$, for which we have the equation (4.4). We rewrite this equation as

$$\begin{aligned}&\varphi'_{k+m,m,l-m}(z,\bar z)\\ &\quad= \frac{1}{2i}\binom{l}{m} g_{kl}(z)(i\langle z,z\rangle)^m - \binom{l-1}{m-1}\langle f_{k+1,l-1}(z),z\rangle(i\langle z,z\rangle)^{m-1} + \cdots, \qquad (7.1)\end{aligned}$$

Where we have included the given term $\varphi_{k+m,m,l-m}(z,\bar z)$ in the dots. Our goal is to write normalization conditions for $\varphi'_{k+m,m,l-m}(z,\bar z)$ that uniquely determine g_{kl} and $f_{k+1,l-1}$. If $m=0$, the term with $f_{k+1,l-1}$ is not present. Hence we have to consider an identity (7.1) with $m \geq 1$. Then the sum of the terms involving g_{kl} and $f_{k+1,l-1}$ is a multiple of $\langle z,z\rangle^{m-1}$. Thus, by varying g_{kl} and $f_{k+1,l-1}$, we may expect to normalize $\varphi'_{k+m,m,l-2m}(z,\bar z)$ to be in the complement of the space of polynomials of the form $P(z,\bar z)\langle z,z\rangle^{m-1}$. The suitable decomposition is given by Proposition 6.1:

$$\varphi'_{k+m,m,l-m}(z,\bar z) = Q(z,\bar z)\langle z,z\rangle^{m-1} + R(z,\bar z), \quad \mathsf{tr}^{m-1}R = 0. \qquad (7.2)$$

Using similar decompositions for other terms in (7.1) and equating the factors of $\langle z,z\rangle^{m-1}$, we obtain

$$Q(z,\bar z) = \frac{1}{2i}\binom{l}{m} g_{kl}(z) i^m\langle z,z\rangle - \binom{l-1}{m-1}\langle f_{k+1,l-1}(z),z\rangle i^{m-1} + \cdots, \qquad (7.3)$$

where, as before, the dots stand for the terms that have been fixed. Since $f_{k+1,l-1}(z)$ is free and the form $\langle z,z\rangle$ is nondegenerate, $\langle f_{k+1,l-1}(z),z\rangle$ is a free bihomogeneous polynomial of the corresponding bidegree. Hence we can normalize Q to be zero. In view of (7.2), $Q=0$ is equivalent to the normalization condition

$$\mathsf{tr}^{m-1}\varphi'_{k+m,m,l-m} = 0. \qquad (7.4)$$

Putting $Q=0$ in (7.3), we can now uniquely solve this equation for $\langle f_{k+1,l-1}(z),z\rangle$ in the form

$$\langle f_{k+1,l-1}(z),z\rangle = \frac{1}{2}\binom{l}{m}\binom{l-1}{m-1}^{-1} g_{kl}(z)\langle z,z\rangle + \cdots, \qquad (7.5)$$

from where $f_{k+1,l-1}(z)$ is uniquely determined. It remains to determine g_{kl}, for which we substitute (7.5) into an identity (7.1) with m replaced by $m' \neq m$:

$$\varphi'_{k+m',m',l-m'}(z,\bar{z}) = \frac{1}{2i}\left(\binom{l}{m'} - \binom{l-1}{m'-1}\binom{l}{m}\binom{l-1}{m-1}^{-1}\right)g_{kl}(z)(i\langle z,z\rangle)^{m'} + \cdots, \quad (7.6)$$

where we have assumed $m' \geq 1$. If $m' = 0$, the term with $f_{k+1,l-1}$ does not occur and hence no substitution is needed. In the latter case, the coefficient in front of $g_{kl}(z)$ is clearly nonzero. Otherwise, for $m' \geq 1$, that coefficient is equal, up to a nonzero factor, to the determinant

$$\begin{vmatrix} \binom{l}{m} & -\binom{l-1}{m-1} \\ \binom{l}{m'} & -\binom{l-1}{m'-1} \end{vmatrix} = \begin{vmatrix} \frac{l!}{m!(l-m)!} & \frac{(l-1)!}{(m-1)!(l-m)!} \\ \frac{l!}{m'!(l-m')!} & \frac{(l-1)!}{(m'-1)!(l-m')!} \end{vmatrix} = \frac{l!(l-1)!}{m!(l-m)!m'!(l-m')!}\begin{vmatrix} 1 & m \\ 1 & m' \end{vmatrix} \neq 0. \quad (7.7)$$

Hence the coefficient in front of $g_{kl}(z)$ in (7.6) is nonzero in any case. Therefore, using Proposition 6.1 as above we see that we can obtain the normalization condition

$$\mathsf{tr}^{m'}\varphi'_{k+m',m',l-m'} = 0, \quad (7.8)$$

which determines uniquely $g_{kl}(z)$ and hence $f_{k+1,l-1}(z)$ in view of (7.5).

Summarizing, for each $m \geq 1$ and $m' \neq m$, we obtain the normalization conditions

$$\mathsf{tr}^{m-1}\varphi'_{k+m,m,l-m} = 0, \quad \mathsf{tr}^{m'}\varphi'_{k+m',m',l-m'} = 0, \quad (7.9)$$

which determine uniquely $g_{kl}(z)$ and $f_{k+1,l-1}(z)$. Such a choice of m and m' is always possible unless $l = 0$. In the latter case, the coefficient $f_{k+1,-1}(z) = 0$ is not present and $g_{k0}(z)$ is uniquely determined by the normalization condition $\varphi'_{k00} = 0$ corresponding to $m' = 0$. That is, for $l = 0$, we only have the second condition in (7.9) with $m' = 0$.

Since the degree of $\varphi'_{k+m,m,l-m}$ is $k + l + m$, we obtain the lowest possible degrees in (7.9) for $m' = 0$, $m = 1$, which corresponds to the normalization

$$\varphi'_{k+1,1,l-1} = 0, \quad \varphi'_{k0l} = 0, \quad (7.10)$$

which is the part of the Chern-Moser normal form [CM74].

7.2. Normalization for $k = 1$

We next analyze the line through $(1,0,l)$ corresponding to the multi-degrees $(m+1, m, l-m)$, $0 \leq m \leq l$, for which we have the equation (4.5). As before, we rewrite this equation as

$$\varphi'_{m+1,m,l-m}(z,\bar z) = \frac{1}{2i}\binom{l}{m} g_{1l}(z)(i\langle z,z\rangle)^m - \binom{l-1}{m-1}\langle f_{2,l-1}(z),z\rangle(i\langle z,z\rangle)^{m-1} - \binom{l-1}{m-1}\langle z,f_{0l}\rangle(-i\langle z,z\rangle)^m + \cdots . \tag{7.11}$$

Arguing as before, we consider $m \ge 1$ and decompose

$$\varphi'_{m+1,m,l-m}(z,\bar z) = Q(z,\bar z)\langle z,z\rangle^{m-1} + R(z,\bar z), \quad \mathsf{tr}^{m-1}R = 0. \tag{7.12}$$

Decomposing similarly the other terms in (7.11) and equating the factors of $\langle z,z\rangle^{m-1}$, we obtain

$$Q(z,\bar z) = \frac{1}{2i}\binom{l}{m} g_{1l}(z) i^m \langle z,z\rangle - \binom{l-1}{m-1}\langle f_{2,l-1}(z),z\rangle i^{m-1} - \binom{l-1}{m-1}\langle z,f_{0l}\rangle(-i)^m\langle z,z\rangle + \cdots . \tag{7.13}$$

Since $f_{2,l-1}(z)$ is free and the form $\langle z,z\rangle$ is nondegenerate, $\langle f_{2,l-1}(z),z\rangle$ is also free and hence we can choose it suitably to obtain $Q=0$, which is equivalent to the normalization condition

$$\mathsf{tr}^{m-1}\varphi'_{m+1,m,l-m} = 0. \tag{7.14}$$

Putting $Q=0$ into (7.13), we solve it uniquely for $\langle f_{2,l-1}(z),z\rangle$, which, in turn, determines uniquely $f_{2,l-1}(z)$. Arguing as in §7.1, we substitute the obtained expression for $\langle f_{2,l-1}(z),z\rangle$ into identities (7.13) with m replaced by $m' \ne m$. The result can be written as

$$\varphi'_{m'+1,m',l-m'}(z,\bar z) = \big(c_{m'}g_{1l}(z) + d_{m'}\langle z,f_{0l}\rangle\big)\langle z,z\rangle^{m'} + \cdots \tag{7.15}$$

with suitable coefficients $c_{m'}$, $d_{m'}$. Then as in §7.1 we see that we can obtain the normalization

$$\mathsf{tr}^{m'}\varphi'_{m'+1,m',l-m'} = 0, \quad \mathsf{tr}^{m''}\varphi'_{m''+1,m'',l-m''} = 0 \tag{7.16}$$

for any pair (m',m'') such that

$$\begin{vmatrix} c_{m'} & d_{m'} \\ c_{m''} & d_{m''} \end{vmatrix} \ne 0. \tag{7.17}$$

Furthermore, assuming (7.17), both $g_{1l}(z)$ and f_{0l} are uniquely determined by (7.16), which, in turn, determine $f_{2,l-1}(z)$ in view of (7.13) with $Q=0$. Thus it remains to choose m', m'' satisfying (7.17). Inspecting the construction of the coefficients $c_{m'}$, $d_{m'}$, it is straightforward to see that the determinant in (7.17) is equal, up to a nonzero multiple, to the determinant

$$\begin{vmatrix} \frac{1}{2i}\binom{l}{m} i^m & -\binom{l-1}{m-1} i^{m-1} & -\binom{l-1}{m-1}(-i)^m \\ \frac{1}{2i}\binom{l}{m'} i^{m'} & -\binom{l-1}{m'-1} i^{m'-1} & -\binom{l-1}{m'-1}(-i)^{m'} \\ \frac{1}{2i}\binom{l}{m''} i^{m''} & -\binom{l-1}{m''-1} i^{m''-1} & -\binom{l-1}{m''-1}(-i)^{m''} \end{vmatrix} \tag{7.18}$$

consisting of the coefficients in (7.13). In fact one can see that the matrix in (7.17) is obtained as a block in (7.19) by elementary row operations. The determinant (7.18) is equal, up to a nonzero multiple, to

$$\begin{vmatrix} 1 & m & (-1)^m m \\ 1 & m' & (-1)^{m'} m' \\ 1 & m'' & (-1)^{m''} m'' \end{vmatrix} = \begin{vmatrix} m'-m & (-1)^{m'}m' - (-1)^m m \\ m''-m & (-1)^{m''}m'' - (-1)^m m \end{vmatrix}$$
$$= \pm \begin{vmatrix} m'-m & (-1)^{m'-m}m' - m \\ m''-m & (-1)^{m''-m}m'' - m \end{vmatrix}. \quad (7.19)$$

In case both $m'-m$ and $m''-m$ are odd, the latter determinant is equal to

$$(m'-m)(-m''-m)-(m''-m)(-m'-m) = 2m''m-2m'm = 2m(m''-m), \quad (7.20)$$

which is nonzero by the construction. In case $m'-m$ is even and $m''-m$ is odd, the last determinant in (7.19) is equal, up to a sign, to

$$(m'-m)(-m''-m)-(m''-m)(m'-m) = 2m''m-2m''m' = 2m''(m-m'), \quad (7.21)$$

which is nonzero provided $m'' \neq 0$. Similarly, in case $m'-m$ is odd and $m''-m$ is even, the determinant is nonzero provided $m' \neq 0$. On the other hand, if $m''=0$, the determinant (7.19) is nonzero if $m'-m$ is odd and similarly, if $m'=0$, the determinant (7.19) is nonzero if $m''-m$ is odd. In all other cases, the determinant is zero.

Summarizing we conclude that (7.19) is nonzero and hence (7.17) holds whenever m, m', m'' are disjoint and the nonzero ones among them are not of the same parity. For such a choice of m, m', m'' with $m \geq 1$, we have the normalization conditions

$$\mathsf{tr}^{m-1}\varphi'_{m+1,m,l-m} = 0, \quad \mathsf{tr}^{m'}\varphi'_{m'+1,m',l-m'} = 0, \quad \mathsf{tr}^{m''}\varphi'_{m''+1,m'',l-m''} = 0, \quad (7.22)$$

that determine uniquely $g_{1l}(z)$, $f_{2,l-1}(z)$ and f_{0l}. Such a choice of m, m', m'' is always possible unless $l \in \{0,1\}$. In case $l=0$, all terms in (7.13) are already zero. If $l=1$, the only choice is $m=1$ and $m'=0$ leaving no space for m''. In that case we still obtain the normalization

$$\varphi'_{101} = 0, \quad \varphi'_{210} = 0, \quad (7.23)$$

which, however, does not determine $g_{11}(z)$, $f_{20}(z)$ and f_{01} uniquely. Instead, we regard f_{01} as a free parameter and then (7.23) determine uniquely $g_{11}(z)$ and $f_{20}(z)$. The free parameter f_{01} corresponds to the choice of $a \in \mathbb{C}^n$ in the following group of automorphisms of the quadric $\mathsf{Im}\, w = \langle z, z \rangle$:

$$(z,w) \mapsto \frac{(z+aw, w)}{1-2i\langle z,a\rangle - i\langle a,a\rangle w}. \quad (7.24)$$

To obtain the lowest possible degrees in (7.22), we have to choose $m=1$, $m'=0$, $m''=0$, in which case (7.22) gives

$$\varphi'_{10l} = 0, \quad \varphi'_{2,1,l-1} = 0, \quad \mathsf{tr}^2\varphi'_{3,2,l-2} = 0, \quad (7.25)$$

which is precisely a part of the Chern-Moser normal form [CM74].

7.3. Normalization for $k = 0$

It remains to analyze the line through $(0,0,l)$ corresponding to the multi-degrees $(m,m,l-m)$, $0 \le m \le l$, for which we have the equation (4.6). Similarly to the above, we rewrite this equation as

$$\begin{aligned}&\varphi'_{m,m,l-m}(z,\bar z)\\&\quad = \binom{l}{m}\mathsf{Im}\left(g_{0l}(i\langle z,z\rangle)^m\right) - 2\binom{l-1}{m-1}\mathsf{Re}\left(\langle f_{1,l-1}(z),z\rangle(i\langle z,z\rangle)^{m-1}\right) + \cdots .\end{aligned} \tag{7.26}$$

In view of the presence of the power i^m inside real and imaginary parts, we have to deal separately with m being even and odd.

We first assume m to be even. In that case (7.26) can be rewritten as

$$\begin{aligned}&\varphi'_{m,m,l-m}(z,\bar z)\\&\quad = \binom{l}{m}(\mathsf{Im}\, g_{0l})(i\langle z,z\rangle)^m - 2i\binom{l-1}{m-1}(\mathsf{Im}\,\langle f_{1,l-1}(z),z\rangle)(i\langle z,z\rangle)^{m-1} + \cdots .\end{aligned} \tag{7.27}$$

Arguing as above we can obtain the normalization

$$\mathsf{tr}^{m-1}\varphi'_{m,m,l-m} = 0, \tag{7.28}$$

implying an equation for $f_{1,l-1}(z)$ and g_{0l} which can be solved for $\mathsf{Im}\,\langle f_{1,l-1}(z),z\rangle$ in the form

$$\mathsf{Im}\,\langle f_{1,l-1}(z),z\rangle = \frac{1}{2}\binom{l-1}{m-1}^{-1}\binom{l}{m}(\mathsf{Im}\, g_{0l})\langle z,z\rangle + \cdots . \tag{7.29}$$

The latter expression is to be substituted into another equation (7.27) with m replaced by m' (still even). As above, we obtain the normalization

$$\mathsf{tr}^{m-1}\varphi'_{m,m,l-m} = 0, \quad \mathsf{tr}^{m'}\varphi'_{m',m',l-m'} = 0, \tag{7.30}$$

provided

$$\begin{vmatrix} \binom{l}{m} & -2\binom{l-1}{m-1} \\ \binom{l}{m'} & -2\binom{l-1}{m'-1}\end{vmatrix} \ne 0, \tag{7.31}$$

which always holds in view of (7.7). Hence $\mathsf{Im}\, g_{0l}$ is uniquely determined by (7.30) and therefore also $\mathsf{Im}\,\langle f_{1,l-1}(z),z\rangle$.

Now assume m is odd. In that case (7.26) becomes

$$\begin{aligned}&\varphi'_{m,m,l-m}(z,\bar z)\\&\quad = \binom{l}{m}(\mathsf{Re}\, g_{0l})i^{m-1}\langle z,z\rangle^m - 2\binom{l-1}{m-1}(\mathsf{Re}\,\langle f_{1,l-1}(z),z\rangle)(i\langle z,z\rangle)^{m-1} + \cdots .\end{aligned} \tag{7.32}$$

Then the above argument yields the normalization

$$\mathsf{tr}^{m-1}\varphi'_{m,m,l-m} = 0, \quad \mathsf{tr}^{m'}\varphi'_{m',m',l-m'} = 0, \tag{7.33}$$

where this time both m and m' are odd. This time (7.33) determines uniquely both $\mathsf{Re}\, g_{0l}$ and $\mathsf{Re}\,\langle f_{1,l-1}(z), z\rangle$.

Summarizing, we obtain the normalization

$$\mathsf{tr}^{m-1}\varphi'_{m,m,l-m} = 0, \qquad \mathsf{tr}^{m'}\varphi'_{m',m',l-m'} = 0,$$
$$\mathsf{tr}^{\widetilde{m}-1}\varphi'_{\widetilde{m},\widetilde{m},l-\widetilde{m}} = 0, \qquad \mathsf{tr}^{\widetilde{m}'}\varphi'_{\widetilde{m}',\widetilde{m}',l-\widetilde{m}'} = 0, \tag{7.34}$$

where $m \geq 1$, $m' \neq m$, $\widetilde{m}' \neq \widetilde{m}$, both m, m' are even and both $\widetilde{m}, \widetilde{m}'$ are odd (note that automatically $\widetilde{m} \geq 1$). The choice of such numbers $m, m', \widetilde{m}, \widetilde{m}'$ is always possible unless $l \in \{0, 1, 2\}$. If $l = 0$, all terms in (7.26) are zero. If $l = 1$, we obtain a valid identity since we have assumed $g_{01} = 1$, $f_{01} = \mathsf{id}$. Finally, for $l = 2$, we can choose $m = 2$, $m' = 0$ for the even part and $\widetilde{m} = 1$ for the odd part but there is no place for $\widetilde{m}'$. We obtain the normalization

$$\varphi'_{002} = 0, \quad \varphi'_{111} = 0, \quad \mathsf{tr}\varphi'_{220} = 0, \tag{7.35}$$

which determines uniquely $\mathsf{Im}\, g_{02}(z)$ and $f_{11}(z)$, whereas $\mathsf{Re}\, g_{02}(z)$ is a free parameter. The latter corresponds to the choice of $r \in \mathbb{R}$ in the following group of automorphisms of the quadric $\mathsf{Im}\, w = \langle z, z\rangle$:

$$(z, w) \mapsto \frac{(z, w)}{1 - rw}. \tag{7.36}$$

To obtain the lowest possible degrees in (7.34), we can choose $m = 2$, $m' = 0$, $\widetilde{m} = 1$, $\widetilde{m}' = 3$, which leads to the normalization

$$\varphi'_{00l} = 0, \quad \varphi'_{1,1,l-1} = 0, \quad \mathsf{tr}\varphi'_{2,2,l-2} = 0, \quad \mathsf{tr}^3\varphi'_{3,3,l-3} = 0, \tag{7.37}$$

which is precisely a part of Chern-Moser normal form [CM74]. However, there is another choice in the lowest degree, namely $m = 2$, $m' = 0$, $\widetilde{m} = 3$, $\widetilde{m}' = 1$, in which case the normalization reads

$$\varphi'_{00l} = 0, \quad \mathsf{tr}\varphi'_{1,1,l-1} = 0, \quad \mathsf{tr}\varphi'_{2,2,l-2} = 0, \quad \mathsf{tr}^2\varphi'_{3,3,l-3} = 0. \tag{7.38}$$

Comparing (7.37) and (7.38) we can say that the Chern-Moser normalization (7.37) has more equations for $\varphi'_{1,1,l-1}$ and less equations for $\varphi'_{3,3,l-3}$. In a sense, the Chern-Moser normalization corresponds to the maximum conditions in the lowest possible degree.

Summarizing the results of this section, we obtain the proof of Theorem 1.1.

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